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# On the Picard group of the stable $\mathbb{A}^1$-homotopy category☆

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**Abstract**

We consider the Picard group Morel and Voevodsky's stable $\mathbb{A}^1$-homotopy category, i.e. the group of objects invertible with respect to the smash product, and show that certain classes of $\mathbb{A}^1$-spectra constructed from Pfister quadrics are elements of this group.


*MSC:* 14F35; 55P42

*Keywords:* Stable $\mathbb{A}^1$-homotopy category; Pfister quadric; Motivic cohomology; Picard group

## 1. Introduction

In this note, we consider the Picard group $Pic(\mathscr{SH}(k))$ of the stable $\mathbb{A}^1$-homotopy category $\mathscr{SH}(k)$ of Morel and Voevodsky, for $k$ any field [22,28]. Throughout this paper, we shall assume that the characteristic of $k$ is not equal to 2. This is the group of objects invertible with respect to the smash product. Our aim is not to calculate the group, as that at present seems to be too difficult a task. Rather, we construct certain examples of elements of $Pic(\mathscr{SH}(k))$. By definition, the one-dimensional projective space $\mathbb{P}^1$ is invertible in $\mathscr{SH}(k)$. The other standard invertible objects are the simplicial circle $S^1_{\mathrm{s}}$ and the twisted circle $S^1_{\mathrm{t}}$. There is a canonical isomorphism in $\mathscr{SH}(k)$

$$S^1_{\mathrm{s}} \wedge S^1_{\mathrm{t}} \cong \mathbb{P}^1$$

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so $S^1_{\mathrm{s}}, S^1_{\mathrm{t}} \in Pic(\mathscr{SH}(k))$ as well. (We shall also use the alternative notation $S^1 = S^1_{\mathrm{s}}$, $S^\alpha = S^1_{\mathrm{t}}$ motivated by Real homotopy theory, see remarks at the end of Section 2.)

By a "new" element of $Pic(\mathscr{SH}(k))$, we mean an invertible object that is not isomorphic to smash products of powers of $S^1_{\mathrm{s}}$ and $S^1_{\mathrm{t}}$ in $\mathscr{SH}(k)$. The examples given in the present paper are as follows. We denote by $Spc(k)_\bullet$ the category of based spaces over a field $k$ (see Section 2 below). For an unbased space $X$ over $k$, the unreduced suspension $\widetilde{X}$ of $X$ is defined by the cofiber sequence in $Spc(k)_\bullet$

$$X_+ \to S^0 \to \widetilde{X},$$

whose first map collapses $X$ to a single point. Let $a \in k^\times/(k^\times)^2$, and $L_a$ be the extension field $k[\sqrt{a}]$. Also, let $S^{\alpha_a} = \mathbb{G}^a_m$ be the affine variety defined by the equation $x^2 - ay^2 = 1$, and let $S^{L_a} = \widetilde{Spec(L_a)}$ be the unreduced suspension of $Spec(L_a)$.

**Proposition 1.1.** *For $a \in k^\times$ not in $(k^\times)^2$, there is a canonical isomorphism in $\mathscr{SH}(k)$*

$$S^{\alpha_a} \wedge S^{L_a} \cong S^{1+\alpha}.$$

**Proposition 1.2.** *For $a, b \in k^\times$, let $Y_{(a,b)}$ denote the projective quadric defined by the homogenous equation $x^2 - ay^2 = bz^2$. Then the unreduced suspension $\widetilde{Y}_{(a,b)}$ is invertible in $\mathscr{SH}(k)$.*

These examples fit into an infinite family of conjectured examples, motivated by an algebraic version of the $\mathbb{Z}/2$-equivariant Hopf invariant one problem, and the "Rost spectra", whose motivic homologies are the Rost motives (see [13,27,28]). For $a_1, \dots, a_n \in k^\times$, the Pfister form defined by $a_1, \dots, a_n$ is the quadratic form

$$\langle\langle a_1, \dots, a_n \rangle\rangle = \otimes_{i=1}^n (x^2 - a_i y^2). \tag{1.3}$$

Define $U_{(a_1,\dots,a_n)}$ to be the affine variety given by the equation

$$\langle\langle a_1, \dots, a_{n-1} \rangle\rangle = a_n.$$

We will see in Section 4 that there are two ways to state algebraic analogues of the Hopf invariant one problem from homotopy theory. Whether or not elements of Hopf invariant one exist, it is reasonable to conjecture that the dimensions in which these elements would lie in the two versions of the problem should agree. We should point out that these "dimensions" are not merely shifts by copies of $S^1_{\mathrm{s}}$ and $S^1_{\mathrm{t}}$, but by $\widetilde{U}_{(a_1,\dots,a_n)}$. This leads to the following conjecture (see Section 4).

**Conjecture 1.4.** *In $\mathscr{SH}(k)$, for $a_1, \dots, a_n \in k^\times$,*

$$U_{(a_1,\dots,a_n,1)} \wedge \widetilde{U}_{(a_1,\dots,a_{n-1},a_n)} \cong \Sigma^{2^{n-1}(1+\alpha)} U_{(a_1,\dots,a_{n-1},1)}.$$

Note that by induction on $n$, this conjecture would imply that $\widetilde{U}_{(a_1,\dots,a_n)}$ and $U_{(a_1,\dots,a_n,1)}$ are in $Pic(\mathscr{SH}(k))$ for all $a_1, \dots, a_n$ in $k^\times$. This is by induction on $n$: Proposition 1.1 gives that $U_{(a,1)}$ is in $Pic(\mathscr{SH}(k))$. If the statement is true for $n-1$, then the right-hand side of Conjecture 1.4 is invertible in $\mathscr{SH}(k)$. Hence, so are the two factors of the left-hand side. The case of $n = 1$ is Proposition 1.1. I also prove the case of $n = 2$, assuming the field $k$ satisfies resolution of singularities in the sense of Hironaka [7]:

**Proposition 1.5.** *For $a, b \in k^{\times}$,*

$$U_{(a,b,1)} \wedge \widetilde{U}_{(a,b)} \cong \Sigma^{2(1+\alpha)} U_{(a,1)}$$

*in $\mathscr{SH}(k)$.*

In particular, this, together with Proposition 1.1, gives that $U_{(a,b,1)}$, i.e. the affine quadric defined by the equation $x^2 - ay^2 - bz^2 + abt^2 = 1$, is in $Pic(\mathscr{SH}(k))$.

The organization of the paper is as follows. In Section 2, I briefly review the basic constructions in $\mathscr{SH}(k)$. In Section 3, I prove Propositions 1.1 and 1.2. I also show that the objects constructed in these propositions give new elements of $Pic(\mathscr{SH}(k))$, not generated by $S^1_{\mathrm{s}}$ and $S^1_{\mathrm{t}}$. In Section 4, I discuss the Hopf invariant one problem in $\mathscr{SH}(k)$, in analogy with the real-oriented Hopf invariant one problem [12]. Finally, the proof of Proposition 1.5 is given in Section 5.

*Added in revision*: Proposition 1.5 needs the full statement of $\mathbb{A}^1$-Atiyah duality for smooth projective varieties. That result is not used elsewhere in the paper. While $\mathbb{A}^1$-Atiyah duality is known (follows for example from Voevodsky's lectures on cross functors [4]), it has, as far as I know, not have been published anywhere. To make the present paper self-contained, a proof is given in Appendix A, which is joint work with I. Kriz (following ideas of Fabien Morel, which can be found in [16]).

## 2. Preliminaries

We begin be recalling some of the basic notions involved in doing homotopy theory in algebraic geometry, due to Morel and Voevodsky [22]. Let $k$ be an arbitrary field, and let $Sm/k$ denote the category of smooth schemes of finite type over $k$. The category $Spc(k)$ of $k$-spaces is defined by adjoining all colimits to $Sm/k$. More specifically, consider the Nisnevich topology on $Sm/k$, which is the subtopology of the étale topology generated by all fundamental squares of the form

$$\begin{array}{ccc} p^{-1}(U) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{i} & X \end{array} \tag{2.1}$$

where $i$ is an open embedding, $p$ is an étale map, and

$$p|_{Y\backslash p^{-1}(U)} : Y\backslash p^{-1}(U) \to X\backslash U$$

is an isomorphism [23]. A presheaf $F$ of sets on $Sm/k$ is a sheaf in the Nisnevich topology if and only if it takes any square of the form (2.1) to a pullback square. Then

$$Spc(k) = \Delta^{op} Sh(Sm/k)_{\mathrm{Nis}}$$

is the category of simplicial sheaves of sets over $Sm/k$ with the Nisnevich topology.

The category $Spc(k)$ is complete and cocomplete, and has a model category structure in which the affine line $\mathbb{A}^1$ plays the role that the unit interval plays in topology. Namely, for a $k$-space $X$, $X \times \mathbb{A}^1$ is a cylinder object for $X$. Let $Spc(k)_{\bullet}$ denote the category of based $k$-spaces, i.e. $k$-spaces with a given map from $Spec(k)$. Let $\mathscr{H}(k)_\bullet$ denote the homotopy category associated with the model structure on $Spc(k)_\bullet$. There are more than one notions of the circle in $Spc(k)_\bullet$. In particular, one has the simplicial circle

$$S^1_{\mathrm{s}} = \mathbb{A}^1/\{0, 1\}$$

and the twisted circle

$$S^1_{\mathrm{t}} = \mathbb{A}^1\backslash\{0\} = \mathbb{G}_m$$

with 1 as the basepoint. Define $\mathbb{T} \in Spc(k)_\bullet$ to be $\mathbb{A}^1/(\mathbb{A}^1\backslash\{0\})$. There are canonical $\mathbb{A}^1$-weak equivalences

$$S^1_{\mathrm{s}} \wedge S^1_{\mathrm{t}} \simeq \mathbb{T} \simeq \mathbb{P}^1. \tag{2.2}$$

There are also canonical $\mathbb{A}^1$-weak equivalences

$$\mathbb{A}^n/(\mathbb{A}^n\backslash\{0\}) \simeq S^{2n,n} = (\mathbb{P}^1)^{\wedge n}$$

for all $n$.

Morel and Voevodsky defined the algebraic stable category by making $\mathbb{P}^1$ invertible under the smash product [22,28]. Namely, a $k$-spectrum is defined to be a sequence of based $k$-spaces $\{E_i\}$, with given structure maps $r_i : \mathbb{P}^1 \wedge E_i \to E_{i+1}$. There is a model structure on the category of $k$-spectra, which is stable in the sense of Bousfield and Friedlander [3]. Let $\mathscr{SH}(k)$ denote the homotopy category associated with this model structure. It is the algebraic analogue of the stable homotopy category in topology. In particular, $\mathscr{SH}(k)$ is a symmetric monoidal category by the smash product [15,11]. Since $\mathbb{P}^1$ is invertible under the smash product in $\mathscr{SH}(k)$ by definition, both $S^1_{\mathrm{s}}$ and $S^1_{\mathrm{t}}$ are invertible in $\mathscr{SH}(k)$.

For $k \subseteq \mathbb{R}$, there is a forgetful functor, complex realization, from $Spc(k)_\bullet$ to the category of based $\mathbb{Z}/2$-equivariant topological spaces, which takes a scheme to its $\mathbb{C}$-points, with complex conjugation as the $\mathbb{Z}/2$-action. Under this forgetful functor, the simplicial circle $S^1_{\mathrm{s}}$ goes to the fixed circle $S^1$, and the twisted circle $S^1_{\mathrm{t}}$ goes to $S^\alpha$, the one-point compactification of the sign representation $\alpha$ of $\mathbb{Z}/2$. Thus, this functor also takes a generalized algebraic cohomology theory to a generalized $\mathbb{Z}/2$-equivariant cohomology theory indexed on the complete $\mathbb{Z}/2$-universe, i.e. all dimensions $k + l\alpha$, $k, l \in \mathbb{Z}$. In the following, we will write $S^1$ for $S^1_{\mathrm{s}}$ and $S^\alpha$ for $S^1_{\mathrm{t}}$ to emphasize this analogy with the $\mathbb{Z}/2$-equivariant category.

We shall deal with many hypersurfaces. In some cases, we will introduce separate notations. In general, however, for an algebraic equation $E$, we will denote by $Sp(E)$ the affine hypersurface defined by $E$, and by $Pr\ (E)$ the projective hypersurface defined by $E$ (provided that $E$ is homogenous).

## 3. Examples of invertible objects in $\mathscr{SH}(k)$

We give the first classes of examples of non-trivial elements of $Pic(\mathscr{SH}(k))$ in this section. Our first goal is to prove Proposition 1.1.

For any object $X$ of the category $Spc(k)$ of (unbased) $k$-spaces, let $X_+ \in Spc(k)_\bullet$ denote $X \amalg Spec(k)$, with $Spec(k)$ as the basepoint. In particular, $S^0 = Spec(k)_+$. For unbased space $X$, let $\widetilde{X}$ denote the unreduced suspension of $X$. For $a \in k^\times$, $a \notin (k^\times)^2$, let $L_a$ denote the extension field $k[\sqrt{a}]$. Let $S^{L_a}$ be the unreduced suspension of $Spec(L_a)$. We also define the affine quadric $\mathbb{G}^a_m$ by

$$\mathbb{G}^a_m = Sp(x^2 - ay^2 = 1). \tag{3.1}$$

The notation $\mathbb{G}_m^a$ is meant to suggest that this is a "twisted" version of the multiplicative group $\mathbb{G}_m$, given by the equation $x^2 - y^2 = 1$.

**Proof of Proposition 1.1.** Consider the join construction in $Spc(k)$, analogous to the join in topology. Given two unbased $k$-spaces $X$ and $Y$, the join $X * Y$ is defined to be the homotopy pushout of the two projections of $X \times Y$ to $X$ and to $Y$. The join has the property that

$$\widetilde{X * Y} \simeq \widetilde{X} \wedge \widetilde{Y}.$$

If $X$ is based, then $\widetilde{X} \simeq \Sigma X$, so the above equation is

$$\Sigma(X * Y) \simeq \Sigma X \wedge \widetilde{Y}.$$

Thus, in the stable category, one gets

$$X * Y \simeq X \wedge \widetilde{Y} \tag{3.2}$$

for based $X$.

Thus, it suffices to show that $\mathbb{P}^1 \simeq \mathbb{G}_m^a * Spec(L_a)$. First, recall the well-known fact that $\mathbb{P}^1$ is isomorphic to the projective quadric $Pr\ (x^2 - ay^2 = z^2)$. In particular, $Spec(L_a)$ embeds into $\mathbb{P}^1$ as the close subvariety given by the equation $x^2 = a$, and $\mathbb{G}_m^a$ is exactly the complement of this in $\mathbb{P}^1$. Consider the open embedding

$$i : \mathbb{G}_m^a \to \mathbb{P}^1$$

and also the map

$$p : Spec(L_a) \times \mathbb{A}^1 \cong (Spec(L_a) \times \mathbb{P}^1) \backslash Spec(L_a) \to \mathbb{P}^1.$$

(Here, the point $Spec(L_a)$ sits in $Spec(L_a) \times \mathbb{P}^1$ via the point at infinity, and $p$ is the restriction of the projection map. Thus, $p$ is an étale map. Then it is straightforward to see that the pullback of these two maps is

$$Spec(L_a) \times \mathbb{G}_m^a$$

and the pullback diagram is

$$\begin{tikzcd} Spec(L_a) \times \mathbb{G}_m^a \arrow[r, "i'"] \arrow[d, "p'"'] & Spec(L_a) \times \mathbb{A}^1 \arrow[d, "p"] \\ \mathbb{G}_m^a \arrow[r, "i"] & \mathbb{P}^1. \end{tikzcd} \tag{3.3}$$

Here, $p'$ is the projection map that collapses $Spec(L_a)$ to $Spec(k)$. One can easily check that $p$ is an isomorphism on the complement of $Spec(L_a) \times \mathbb{G}_m^a$ in $Spec(L_a) \times \mathbb{A}^1$, which is just $Spec(L_a)$. So the square (3.3) is of the form (2.1), and is therefore a pushout square in $Spc(k)$. Also, the projection map $Spec(L_a) \times \mathbb{A}^1 \to Spec(L_a)$ is an $\mathbb{A}^1$-homotopy equivalence by contracting $\mathbb{A}^1$. It is easy to check that the map $i'$, composed with this projection map, is the projection of $Spec(L_a) \times \mathbb{G}_m^a$ onto $Spec(L_a)$. Thus, $\mathbb{P}^1$ is the homotopy pushout of the projections from $Spec(L_a) \times \mathbb{G}_m^a$ to $Spec(L_a)$ and to $\mathbb{G}_m^a$, i.e. the join of $Spec(L_a)$ and $\mathbb{G}_m^a$. $\square$

**Proposition 3.4.** *For $a \notin (k^\times)^2$, $\mathbb{G}_m^a$ is not generated by $S^1_{\mathrm{s}}$ and $S^1_{\mathrm{t}}$ via the smash product in $\mathscr{SH}(k)$.*

**Proof.** Consider the cofiber sequence

$$Spec(L_a)_+ \to S^0 \to \widetilde{Spec(L_a)} = S^{L_a}. \tag{3.5}$$

If $k$ is a perfect field, this gives rise to a distinguished triangle in $DM$, the derived category of motives, which relates the motives of $Spec(L_a)_+$, $Spec(k)_+$ and $\widetilde{Spec(L_a)}$. Using this distinguished triangle, we see that there are no non-trivial maps from Tate motives of non-negative weight to $M(\widetilde{Spec(L_a)})$, the motive of $\widetilde{Spec(L_a)}$. So $M(\widetilde{Spec(L_a)})$ is not a Tate motive, and is not in the tensor subcategory of $DM$ generated by the motives of $S^1_{\mathrm{s}}$ and $S^1_{\mathrm{t}}$. For general $k$ of finite characteristic $\neq 2$, there is an (infinite) algebraic field extension $k_{perf}$ of $k$ so that $k_{perf}$ is perfect, and the degree of $k_{perf}$ over $k$ is relatively prime to 2. Then $k^\times/(k^\times)^2$ maps injectively into $k^\times_{perf}/(k^\times_{perf})^2$, and the above argument holds over $k_{perf}$. So $S^{L_a} = \widetilde{Spec(L_a)}$ is not stably equivalent to $S^{s+t\alpha}$ for any $s, t$. Therefore, neither is $S^{\alpha_a} \simeq \mathbb{G}_m^a$. $\square$

**Remark.** In the case $a \in (k^\times)^2$, we can still make sense of the statement of the proposition by replacing $Spec(L_a)$ by $S^0 = Spec(k)_+$. Then $\mathbb{G}_m^a \cong \mathbb{G}_m = S^1_{\mathrm{t}}$, and $S^{L_a} \cong S^1_{\mathrm{s}}$. In general, the pairs $\mathbb{G}_m^a$, $S^{L_a}$ are parametrized by $k^\times/(k^\times)^2$. For instance, consider $k \subseteq \mathbb{R}$. For the purpose of taking complex realizations into the $\mathbb{Z}/2$-equivariant category, we can assume that $a = 1$ or $-1$. If $a = 1$, then $\mathbb{G}_m^a = \mathbb{G}_m$, which goes to $S^\alpha$ under complex realization. On the other hand, if $a = -1$, then $Spec(L_a) = \{i, -i\}$ goes to $\mathbb{Z}/2$ under complex realization. So the complex realization of $S^{L_a}$ is $\widetilde{\mathbb{Z}/2} = S^\alpha$, and the complex realization of $\mathbb{G}_m^a \simeq S^{2,1} \wedge (S^{L_a})^{-1}$ is $S^{2,1} \wedge (S^1_{\mathrm{t}})^{-1} \simeq S^1$ for $a = -1$.

We can also generalize the above example in a certain sense. Consider the projective quadric $Pr\,(x^2 - ay^2 = bz^2)$ as a generalization of $\mathbb{P}^1 \cong Pr\,(x^2 - ay^2 = z^2)$. Proposition 1.2 states that its unreduced suspension is invertible in $\mathscr{SH}(k)$. From now on, for a vector bundle $\xi$ with total space $E$ over an algebraic variety $X$, let

$$X^\xi = E/(E \backslash X).$$

This will be called the *Thom space* of $\xi$.

We shall need the following theorem of Morel–Voevodsky ([22], Theorem 3.2.23). We give an alternate proof here for completeness.

**Lemma 3.6.** *Let $Z$ be a smooth affine variety and let $Y \subset Z$ be a closed subvariety; let $\xi$ be the normal bundle of $Y$ in $Z$. Then we have an $\mathbb{A}^1$-homotopy equivalence*

$$\varphi : Z/(Z\backslash Y) \simeq Y^\xi.$$

*Furthermore, we have a commutative diagram in the $\mathbb{A}^1$-homotopy category*

$$\begin{tikzcd} X/(X\backslash(Y\cap X)) \arrow[r, "\varphi", "\simeq"'] \arrow[d] & (Y\cap X)^\xi \arrow[d] \\ Z/(Z\backslash Y) \arrow[r, "\varphi", "\simeq"'] & Y^\xi \end{tikzcd}$$

*where the vertical maps are inclusions, if either $X$ is an open affine subvariety of $Z$, or a closed subvariety of $Z$ such that $X, Y$ intersect transversally in $Z$.*

**Proof.** First of all, since $Y$ is smooth, it is a locally complete intersection, so it can be covered by Zariski open sets $U_i$ in $Z$, such that for each $i$, $Y \cap U_i$ is a complete intersection in $U_i$, i.e. there are functions $f_{i1}, \ldots, f_{ik} : U_i \to \mathbb{A}^1$, such that the function

$$f_i = (f_{i1}, \ldots f_{ik}) : U_i \to \mathbb{A}^k$$

is transverse to 0, and $Y \cap U_i$ is the zero locus of $f_i$.

By further refining $\{U_i\}$, we may also assume that for each $i$, there exists a linear surjective map

$$\varphi_i : \mathbb{A}^n \to \mathbb{A}^{n-k}$$

such that $\varphi_i|_{U_i \cap Y}$ is étale. We have the following claim.

**Claim.** *There exists a map*

$$g_i : U_i/(U_i \backslash (U_i \cap Y)) \cong ((U_i \cap Y) \times \mathbb{A}^k)/((U_i \cap Y) \times (\mathbb{A}^k \backslash \{0\}))$$

*which is the identity on $U_i \cap Y$.*

**Proof.** Recall that we have

$$(U_i \cap Y) \times_{\mathbb{A}^{n-k}} (U_i \cap Y) = \amalg_q (U_i \cap Y)$$

for some number $q$ (the pullback is along the map $\varphi_i$ on both sides). Hence, we can form a Nisnevich square

$$\begin{tikzcd}
(U_i \backslash (U_i \cap Y)) \times_{\mathbb{A}^{n-k}} (U_i \cap Y) \arrow[r, "\subset"] \arrow[d] & (U_i \times_{\mathbb{A}^{n-k}} (U_i \cap Y)) \backslash \amalg_{q-1} (U_i \cap Y) \arrow[d] \\
U_i \backslash (U_i \cap Y) \arrow[r] & U_i .
\end{tikzcd} \tag{3.7}$$

Therefore, $U_i/(U_i \backslash (U_i \cap Y))$ is isomorphic to the quotient of the top row of (3.7). But we also have another Nisnevich square

$$\begin{tikzcd}
(U_i \backslash (U_i \cap Y)) \times_{\mathbb{A}^{n-k}} (U_i \cap Y) \arrow[r, "\subset"] \arrow[d, "f_i \times Id"'] & (U_i \times_{\mathbb{A}^{n-k}} (U_i \cap Y)) \backslash \amalg_{q-1} (U_i \cap Y) \arrow[d, "f_i \times Id"] \\
(\mathbb{A}^k \backslash \{0\}) \times (U_i \cap Y) \arrow[r] & \mathbb{A}^k \times (U_i \cap Y).
\end{tikzcd} \tag{3.8}$$

Hence, the quotient of the top row of (3.7) is also isomorphic to the quotient of the bottom row of (3.8), concluding the proof of the claim. □

Now for a finite sequence of indices $I$, set $U_I = \cap_{i \in I} U_i$. Given the maps $g_i$, we will now produce a map

$$\mathrm{hocolim}_I U_I/(U_I \backslash (U_I \cap Y)) \to Y^{\xi}. \tag{3.9}$$

Here, by the left-hand side, we mean the simplicial realization of the simplicial $\mathbb{A}^1$-space whose $k$th stage is

$$\amalg_{|I|=k+1} U_I/(U_I\backslash(U_I \cap Y)).$$

Standard arguments then imply that this is equivalent to $Z/(Z\backslash Y)$.

To get the map (3.9), note that the $U_i$ trivialize the algebraic bundle $\xi$, so on a vertex $U_i/(U_i\backslash(U_i \cap Y))$ of the simplicial $\mathbb{A}^1$-space, we can use the map $g_i$. It remains to extend them to the higher simplices. Note that if we have a set of choices $g_{i1}, \ldots, g_{il}$ of the map $g_i$ for a fixed $i$, which are given by our construction in the proof of the previous claim, so that the choices of the map $f_i$ have the same derivative at $U_i \cap Y$, then any affine linear combination $s_1 g_{i1} + \cdots + s_l g_{il}$, where $\sum s_j = 1$, also satisfies the statement of our previous claim. But now we can choose the transition functions of $\xi$ to be precisely the $(Df_i)^{-1} Df_j$. Then by induction on the intersections of the $U_i$'s, the equal derivative hypothesis for the $g_i$'s, when composed with the appropriate transition functions, is always satisfied. Therefore, the maps $g_i$ can be extended to (3.9) by affine linear combinations.

The functoriality stated is obtained in the obvious way by intersecting the $U_i$'s with $X$. □

**Proof of Proposition 1.2.** For fixed $a, b \in k^\times$, write $Y$ for $Y_{(a,b)} = Pr\ (x^2 - ay^2 = bz^2)$. We can assume that $a$ and $b$ are not in $(k^\times)^2$. We also recall the well-known fact that $Y$ has the property that $Y \times Y$ is a $\mathbb{P}^1$-bundle over $Y$.

Now consider the diagonal embedding

$$\Delta : Y_+ \to (Y \times Y)_+.$$

For any variety $X$, the normal bundle of the diagonal is isomorphic to the tangent bundle of $X$. In effect, the normal space to a point $(P, P)$ is isomorphic to the tangent space at $P$ by projection to the second coordinate. So we have a cofiber sequence

$$((Y \times Y)\backslash\Delta(Y))_+ \to (Y \times Y)_+ \to (Y \times Y)/((Y \times Y)\backslash\Delta(Y)) \simeq Y^\tau, \tag{3.10}$$

where $Y^\tau$ denotes the Thom space of the tangent bundle $\tau$ of $Y$. However, consider the map

$$\pi_1 : (Y \times Y)\backslash\Delta(Y) \to Y,$$

which is the projection onto the first variable, restricted to $(Y \times Y)\backslash\Delta(Y)$. Then $\pi_1$ is the projection map of an affine bundle, and hence a weak equivalence. Thus, we have a diagram in $\mathscr{H}(k)_\bullet$

$$\begin{tikzcd} ((Y \times Y)\backslash\Delta(Y))_+ \arrow[r] \arrow[d, "\pi_1", "\simeq"'] & (Y \times Y)_+ \arrow[r, "g"] & Y^\tau \\ Y_+ \arrow[ur, "f"'] & & \end{tikzcd}.$$

So we in fact have a cofiber sequence

$$Y_+ \xrightarrow{f} (Y \times Y)_+ \xrightarrow{g} Y^\tau. \tag{3.11}$$

For a bundle $\xi$ over $Y$, consider the pullback $\pi_1^*\xi$ on $Y \times Y$ of $\xi$ with respect to projection onto the first coordinate. We have $(Y \times Y)^{\pi_1^*\xi} \simeq Y^\xi \wedge Y_+$. Here, $X^\eta$ denotes the Thom space of a vector bundle on $X$:

if the total space of the bundle is $E$, then $X^{\eta} = E/(E\backslash X)$, where $X$ is identified with the 0-section of the bundle. Note that the restriction of $\pi_1^*\xi$ on $Y \times Y$ to $\Delta(Y)$ is just $\xi$ itself, so the map $g$ is covered by a map of bundles from $\pi_1^*\xi$ to $\xi$.

From now on, we will repeatedly use the fact that the process of taking Thom spaces (which we will call simply *Thomification*) is functorial with respect to morphisms of bundles (for which we require to be iso on fibers). It is also convenient to consider *based Thomification*, by which we shall mean, for a bundle $\xi$ over a based space $X$, the space $X^{\xi}_{\bullet}$ obtained from $X^{\xi}$ by collapsing the fiber over the base point to a point (a slightly different formulation using pairs of spaces will also be considered below). Now *based* Thomification preserves cofiber sequences. This is true both in classical and $\mathbb{A}^1$-homotopy theory. In $\mathbb{A}^1$-homotopy theory, a cofiber of an inclusion can be defined as a quotient sheaf in the Nisnevich (or alternately cd-h) topology, which passes to a quotient of an inclusion of sheaves after taking based Thomification.

For example, taking Thom spaces with respect to $\xi$ on (3.10) gives

$$((Y \times Y)\backslash\Delta(Y))^{\pi_1^*\xi} \to (Y \times Y)^{\pi_1^*\xi} \xrightarrow{g^{\xi}} Y^{\xi\oplus\tau}. \tag{3.12}$$

Note that the map $\pi_1$ induces an $\mathbb{A}^1$-equivalences of Thom spaces

$$\tilde{\pi}_1 : ((Y \times Y)\backslash\Delta(Y))^{\pi_1^*\xi} \to Y^{\xi}$$

and we have the commutative diagram

$$\begin{tikzcd}
((Y \times Y)\backslash\Delta(Y))^{\pi_1^*\xi} \arrow[r] \arrow[d, "\pi_1", "\simeq"'] & (Y \times Y)^{\pi_1^*\xi} \arrow[d, "\pi_1"] \\
Y^{\xi} \arrow[r, "="] & Y^{\xi}.
\end{tikzcd} \tag{3.13}$$

Now in (3.12), we can replace $((Y \times Y)\backslash\Delta(Y))^{\pi_1^*\xi}$ by $Y^{\xi}$, giving the cofiber sequence

$$Y^{\xi} \xrightarrow{f^{\xi}} Y^{\xi} \wedge Y_+ \xrightarrow{g^{\xi}} Y^{\tau\oplus\xi}. \tag{3.14}$$

We will show that the cofiber sequence (3.14) also holds stably if $\xi$ is a virtual bundle, i.e. an element of $K^0Y$. To this end, one must define Thomification for virtual bundles. Recall that if $\xi$ is a virtual bundle over an affine variety $U$, then we always have

$$\xi \oplus n = \zeta,$$

where $\zeta$ is an actual bundle, and $n \geqslant 0$ (in the formula, $n$ denotes the trivial bundle of that dimension). Thus, one can just desuspend the cofiber sequence for $\zeta$ by $S^{n(1+\alpha)}$. Here, $Y$ is a projective variety. However, for any projective variety $M$, we can define an affine variety $U(M)$ which is $\mathbb{A}^1$-weak equivalent to $M$. For $M = \mathbb{P}^m$, $U(\mathbb{P}^m) \subset \mathbb{P}^m \times \mathbb{P}^m$ is the complement of the projective subvariety given by the equation

$$\sum_{i=0}^{m} x_i y_i = 0,$$

where $x_i$ and $y_i$ are the projective coordinates in the two copies of $\mathbb{P}^m$, respectively. Projection onto the first coordinate $U(\mathbb{P}^m) \to \mathbb{P}^m$ is easily seen to be an $\mathbb{A}^m$-bundle, and thus an $\mathbb{A}^1$-weak equivalence in $Spc(k)$. For general $M$,

$$U(M) \to M \tag{3.15}$$

is just the pullback of this bundle via an inclusion $M \to \mathbb{P}^m$.

The case we will be most interested in is the Thom spectrum $Y^{\nu_Y}$ where $Y$ is a smooth projective variety. By this, we shall mean the Thom spectrum $U(Y)^\mu$, where $\mu$ is the complement of the pullback of the tangent bundle of $Y$ to $U(Y)$.

Now consider the particular $Y$ defined at the beginning of this proof, and denote $\nu = \nu_Y$. By (3.14), we have a cofiber sequence

$$Y^\nu \xrightarrow{f^\nu} Y^\nu \wedge Y_+ \xrightarrow{g^\nu} Y^{\nu \oplus \tau} = Y_+.$$

By the diagram (3.13), the map $f^\nu : Y^\nu \to Y^\nu \wedge Y_+$ is split by

$$\pi_1 : Y^\nu \wedge Y_+ \simeq (Y \times Y)^{\pi_1^* \nu} \to Y^\nu.$$

The map $\pi_1$ collapses $Y_+$ to $S^0$, so its stable fiber is $\Sigma^{-1} Y^\nu \wedge \widetilde{Y}$, so we have an equivalence

$$\varphi : Y^\nu \wedge \widetilde{Y} \simeq \Sigma Y_+$$

since $Y_+$ is the stable cofiber of $f^\nu$. Let $j$ be the map constructed in Lemma 3.18 below. We will show that the following square commutes in the stable $\mathbb{A}^1$-homotopy category:

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "j \wedge Id"] \arrow[d, "="'] & Y^\nu \wedge \widetilde{Y} \arrow[d, "\simeq"', "\varphi"] \\ \widetilde{Y} \arrow[r, "\delta"'] & \Sigma Y_+. \end{tikzcd} \tag{3.16}$$

(Here, $\delta$ denotes the connecting map.) Granted that, the stable fiber of the top map is $F(j) \wedge \widetilde{Y}$ (where $F(j)$ denotes the fiber of $j$), and the stable fiber of the bottom map is $S^0$, and the two are equivalent. This shows that $\widetilde{Y}$ is invertible in the stable $\mathbb{A}^1$-homotopy category and that its inverse is $F(j)$. (And hence $F(j) \simeq D\widetilde{Y}$, where $D\widetilde{Y}$ denotes the Spanier–Whitehead dual of $\widetilde{Y}$. By definition, the Spanier–Whitehead dual $DX$ of a spectrum $X$ is the function spectrum $F(X, S^0)$, also called the $S$-dual, see [11].)

To show that the square (3.16) commutes up to homotopy, consider the diagram

$$\begin{tikzcd} \Sigma^{-1}\widetilde{Y} \arrow[r, "\delta"] \arrow[d, "j \wedge Id"'] & Y_+ \arrow[d, "j \wedge Id"] \\ Y^\nu \wedge \Sigma^{-1}\widetilde{Y} \arrow[r, "\delta"] \arrow[d, "\Sigma^{-1}\varphi"', "\simeq"] & Y^\nu \wedge Y_+ \arrow[d, "="'] \\ Y_+ & Y^\nu \wedge Y_+. \arrow[l, "g^\nu"'] \end{tikzcd}$$

It is easy to check that this diagram commutes up to homotopy: the top square commutes by the naturality of $\delta$, and the bottom square commutes by the definition of $\varphi$. It suffices to show that the composition

$$g^{\nu} \circ (j \wedge id) : Y_{+} \to Y^{\nu} \wedge Y_{+} \to Y_{+} \tag{3.17}$$

is homotopic to the identity. But that is part of the statement of Lemma 3.18 below. □

The preceding proof relied on the following result.

**Lemma 3.18.** *Let $Y$ be a smooth projective variety over $k$, and let $\nu$ be the virtual normal bundle of $Y$* (*see definition after* (3.14)). *Then there exists a map* (*in the stable $\mathbb{A}^1$-homotopy category*)

$$j : S^0 \to Y^{\nu} \tag{3.19}$$

*such that the composition*

$$g^{\nu} \circ (j \wedge Id) : Y_{+} \to Y^{\nu} \wedge Y_{+} \to Y_{+}$$

*is stably $\mathbb{A}^1$-homotopic to the identity*.

*Comment*: The map $j$ is similar to the map constructed in Proposition 2.7 of Voevodsky [30], but I have not checked that the two constructions are equivalent.

**Proof of Lemma 3.18.** First, for any smooth projective variety $Y$, denote by $\varepsilon$ the composition

$$Y^{\nu} \wedge Y_{+} \xrightarrow{g^{\nu}} Y_{+} \to S^0. \tag{3.20}$$

Then we see that in turn, $g^{\nu}$ is the composition

$$Y^{\nu} \wedge Y_{+} \xrightarrow{Id \wedge \Delta} Y^{\nu} \wedge Y_{+} \wedge Y_{+} \xrightarrow{\varepsilon \wedge Id} Y_{+}.$$

Therefore, it suffices to construct a map $j$ such that the composition

$$Y_{+} \xrightarrow{j \wedge Id} Y^{\nu} \wedge Y_{+} \xrightarrow{\varepsilon} S^0 \tag{3.21}$$

is the collapse map.

Note also that by adjunction, the map $\varepsilon$ gives a map

$$\lambda_Y : Y^{\nu} \to F(Y_{+}, S^0) = DY_{+} \tag{3.22}$$

(recall from above that $DY_{+}$ denotes the Spanier–Whitehead dual of $Y_{+}$). In the Appendix A, it will be shown that this map is an equivalence, but in the proof of Proposition 1.2, I only use the case of this fact for projective spaces. It is worth noting that the map (3.22) is functorial with respect to closed immersions $i : X \subseteq Y$: let $\mu$ be the normal bundle of $X$ in $Y$. We have

$$Y/(Y \backslash X) \simeq X^{\mu}. \tag{3.23}$$

(Consider the affinized embedding $U(X) \subseteq U(Y)$, and use Lemma 3.6.) Then we also have

$$\nu_X = \nu_Y \oplus \mu$$

as virtual bundles, so we have a diagram

$$\begin{tikzcd} Y^{\nu_Y} \arrow[r, "\lambda_Y"] \arrow[d, "q_X^Y"'] & DY_+ \arrow[d, "Di"] \\ X^{\nu_X} \arrow[r, "\lambda_X"'] & DX_+ \end{tikzcd} \tag{3.24}$$

where $q_X^Y$ is the $\nu_Y$-Thomification of (3.23). The diagram (3.24) commutes. Indeed, this is adjoint to the commutativity of the diagram

$$\begin{tikzcd} Y^{\nu_Y} \wedge X_+ \arrow[r, "Id\wedge i_+"] \arrow[d, "q_X^Y"'] & Y^{\nu_Y} \wedge Y_+ \arrow[d, "g^\nu"] \\ X^{\nu_X} \wedge X_+ \arrow[r, "i_+\cdot g^\nu"'] & Y_+ \end{tikzcd}$$

which is, by construction, the $\nu_Y$-Thomification of the commutative diagram

$$\begin{tikzcd} (Y \times X)_+ \arrow[r, "i"] \arrow[d] & (Y \times Y)_+ \arrow[dd] \\ (Y \times X)/((Y \times X)\backslash(X \times X)) \arrow[d] & \\ (Y \times X)/((Y \times X)\backslash\Delta(X)) \arrow[r] & (Y \times Y)/((Y \times Y)\backslash\Delta(Y)) \end{tikzcd}$$

(the unlabeled arrows are projections).

Next, consider the composition map

$$(Y\backslash X) \times Y \to Y \times Y \to (Y \times Y)/(Y \times Y\backslash\Delta(Y)). \tag{3.25}$$

Then, since $(Y\backslash X) \times X$ is mapped to $(Y \times Y)\backslash\Delta(Y)$, (3.25) induces a map

$$((Y\backslash X) \times Y)/((Y\backslash X) \times X) \to (Y \times Y)/((Y \times Y)\backslash\Delta(Y)) \tag{3.26}$$

or, as above

$$(Y\backslash X)^{\nu_Y} \wedge (Y/X) \to Y_+. \tag{3.27}$$

Composing with the collapse map $Y_+ \to S^0$ and taking the adjoint, we get a map

$$\lambda_{Y/X} : (Y\backslash X)^{\nu_Y} \to DY/X.$$

Moreover, by construction, we obtain a commutative diagram in the stable $\mathbb{A}^1$-homotopy category

$$\begin{tikzcd} (Y\backslash X)^{\nu_Y} \arrow[r, "\lambda_{Y/X}"] \arrow[d, "j^{\nu_Y}"'] & DY/X \arrow[d, "Dp"] \\ Y^{\nu_Y} \arrow[r, "\lambda_Y"'] & DY_+ \end{tikzcd} \tag{3.28}$$

where $j : Y\backslash X \to Y$ is the inclusion map, and $p : Y_+ \to Y/X$ is the projection.

**Claim 1.** *The diagrams* (3.24) *and* (3.28) *together give a map of distinguished triangles formed by the vertical maps*.

**Proof.** Let $M_1$, $M_2$ be the mapping cylinders of the inclusions $Y\backslash X \to Y$, $X \to Y$ respectively. Consider the canonical inclusion of $\Delta(Y) \times \mathbb{A}^1 \times \mathbb{A}^1$ in $(Y \times \mathbb{A}^1) \times (Y \times \mathbb{A}^1)$. We also have an inclusion $M_1 \times M_2 \subset (Y \times \mathbb{A}^1) \times (Y \times \mathbb{A}^1)$. Let

$$\widetilde{\Delta} = (\Delta(Y) \times \mathbb{A}^1 \times \mathbb{A}^1) \cap (M_1 \times M_2).$$

Then $\widetilde{\Delta}$ is the mapping cylinder of

$$i \amalg j : X \amalg (Y\backslash X) \to Y. \tag{3.29}$$

Define

$$\Delta = \widetilde{\Delta}/\widetilde{\Delta} \cap ((M_1 \times \{1\}) \cup (\{1\} \times M_2));$$

this is isomorphic to the mapping cone of (3.29).

Now by Lemma 3.6, we have an equivalence

$$(M_1 \times M_2)/((M_1 \times M_2\backslash\widetilde{\Delta}) \cup ((M_1 \times \{1\}) \cup (\{1\} \times M_2)) \simeq \Delta^{\tau_Y}, \tag{3.30}$$

where $\Delta^{\tau_Y}$ is the cofiber of the Thomification

$$i^{\tau_Y} \amalg j^{\tau_Y} : X^{\tau_Y} \amalg (Y\backslash X)^{\tau_Y} \to Y^{\tau_Y}.$$

Let $C_1$, $C_2$ be the mapping cones of the inclusions $Y\backslash X \to Y$, $X \to Y$ respectively. In particular, (3.30) induces a map in the unstable $\mathbb{A}^1$-homotopy category

$$C_1 \wedge C_2 \to \Delta^{\tau_Y}. \tag{3.31}$$

Now there is a functor of based Thomification from the category of pairs of spaces $(X, A)$ with a bundle $\eta$ on $X$ to based spaces, which takes $(X, A)$ to $X^\eta/A^\xi$, where $\xi$ is the pullback of $\eta$ to $A$. Passing to affinizations, and using this based Thomification, (3.31) induces a map

$$X^{\nu_X} \wedge Y/X \to \Delta. \tag{3.32}$$

Now to prove Claim 1, we must prove the commutativity of the diagram

$$\begin{tikzcd} X^{\nu_X} \arrow[r, "\lambda_X"] \arrow[d, "\partial"'] & DX_+ \arrow[d, "\partial"] \\ \Sigma(Y\backslash X)^{\nu_Y} \arrow[r, "\Sigma\lambda_{Y/X}"'] & \Sigma DY/X \end{tikzcd} \tag{3.33}$$

where the vertical maps are the connecting maps of the distinguished triangles considered.

As before, we consider the adjoint diagram; after suspension, this diagram consists of two maps

$$X^{\nu_X} \wedge Y/X \to S^1.$$

However, examining the constructions involved, we see that these two maps are homotopic in the stable $\mathbb{A}^1$-category to the compositions of (3.32) with the two collapse maps

$$\Delta \to S^1$$

obtained by collapsing the cone on $X$ (resp. $Y \backslash X$) to the basepoint and projecting the remaining cone to its suspension coordinate. These maps, however, are homotopic: this is equivalent to taking a model of $S^1$ by attaching two copies of $\mathbb{A}^1$ at two points (say, 0, 1), and saying that the two maps into $\mathbb{A}^1/\{0, 1\}$ obtained by collapsing either copy of $\mathbb{A}^1$ into a point (with the right orientation) are homotopic. This, however, is equivalent to the corresponding fact about simplicial sets, which is standard. $\square$

Now our next step is to prove the following claim.

**Claim 2.** *For $Y = \mathbb{P}^n$, the map $\lambda_{\mathbb{P}^n}$ of* (3.22) *is an equivalence*.

**Proof.** By induction on $n$. For $n = 0$, the statement is obvious. Suppose the statement is true for $n - 1$. Consider the embedding $i : X \subseteq Y$, where $X = \mathbb{P}^{n-1}$ and $Y = \mathbb{P}^n$. Then Claim 1 gives a morphism of distinguished triangles

$$\begin{tikzcd}
S^{-n(1+\alpha)} \arrow[r, "\simeq"] \arrow[d] & S^{-n(1+\alpha)} \arrow[d] \\
(\mathbb{P}^n)^{v_{\mathbb{P}^n}} \arrow[r, "\lambda_{\mathbb{P}^n}"] \arrow[d, "q^{\mathbb{P}^n}_{\mathbb{P}^{n-1}}"'] & D\mathbb{P}^n_+ \arrow[d, "Di"] \\
(\mathbb{P}^{n-1})^{v_{\mathbb{P}^{n-1}}} \arrow[r, "\lambda_{\mathbb{P}^{n-1}}"'] & D\mathbb{P}^{n-1}_+ .
\end{tikzcd}$$

To see that the top row is an equivalence, recall that it is the adjoint of (3.27) which, by construction, is in this case just the duality

$$S^{-n(1+\alpha)} \wedge S^{n(1+\alpha)} \to S^0$$

(as $\mathbb{P}^n - \mathbb{P}^{n-1} \cong \mathbb{A}^n$, $\mathbb{P}^n/\mathbb{P}^{n-1} \cong S^{n(1+\alpha)}$.) Now since $\lambda_{\mathbb{P}^{n-1}}$ is an $\mathbb{A}^1$-equivalence, by the induction hypothesis, so is $\lambda_{\mathbb{P}^n}$. $\square$

Now by the Claim, we can construct $j$ for $Y = \mathbb{P}^n$ as the composition

$$j : S^0 \to D\mathbb{P}^n_+ \xrightarrow{\lambda^{-1}_{\mathbb{P}^n}} (\mathbb{P}^n)^{v_{\mathbb{P}^n}}. \tag{3.34}$$

Here, the first map is the dual of the collapse map. For an arbitrary smooth projective $Y$, choose an embedding

$$Y \subseteq \mathbb{P}^n$$

and define $j$ as the composition

$$S^0 \xrightarrow{j_{\mathbb{P}^n}} (\mathbb{P}^n)^{v_{\mathbb{P}^n}} \xrightarrow{q^{\mathbb{P}^n}_Y} Y^{v_Y}.$$

To prove the statement that (3.21) is equivalent to the collapse map, in adjoint form it says that the composition

$$S^0 \xrightarrow{j} Y^{\nu_Y} \xrightarrow{\lambda_Y} DY_+$$

is the dual of the collapse map. By the naturality (3.24), this only needs to be proven for $Y = \mathbb{P}^n$, where it is true by definition. □

**Corollary 3.35.** *For the affine quadric $U_{(a,b)} = Sp(x^2 - ay^2 = b)$, the unreduced suspension $\widetilde{U}_{(a,b)}$ is invertible in $\mathscr{SH}(k)$.*

**Proof.** Note that the affine quadric $U = Sp(x^2 - ay^2 = b)$ is the complement of $Spec(k[\sqrt{a}]) = Pr\ (x^2 - ay^2 = 0)$ in $Pr\ (x^2 - ay^2 = bz^2)$. So by methods similar to the proof of Proposition 1.1, we have

$$Sp(x^2 - ay^2 = b) * (Spec(k[\sqrt{a}]) \simeq Pr\ (x^2 - ay^2 = bz^2).$$

Hence,

$$\widetilde{U}_{(a,b)} \wedge S^{L_a} \simeq \widetilde{Y}_{(a,b)}.$$

So in $\mathscr{SH}(k)$, $\widetilde{U}_{(a,b)} = \widetilde{Y}_{(a,b)} \wedge (S^{L_a})^{-1}$ is invertible. □

**Proposition 3.36.** *In $Pic(\mathscr{SH}(k))$, $\widetilde{Y}_{(a,b)}$ is not generated by $S^1$ and $S^\alpha$.*

**Proof.** Recall that we are assuming $char(k) \neq 2$. Also, by an analogous argument as in the proof of Proposition 3.4, we may assume that $k$ is perfect. Now the projective quadric $Pr\ (x^2 - ay^2 = bz^2)$ is just one case of Pfister quadrics of the form

$$Y_{(a_1,\ldots,a_n)} = Pr\ (\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle = a_n z^2),$$

where $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ is the Pfister form associated with the elements

$$a_1, \ldots, a_{n-1} \in k^\times$$

given by (1.3). The cohomology of Pfister quadrics has been calculated by Rost [27]. For $(a, b) \neq 0$ in the mod 2 Milnor $K$-theory $K_*^M(k)/2$ of $k$, the mod 2 motivic cohomology of $(Y_{(a,b)})_+$ is not the same as that of $\mathbb{P}^1_+$. Thus, if $(a, b) \neq 0$ in $K_2^M(k)/2$, $(Y_{(a,b)})_+$ is not stably equivalent to $\mathbb{P}^1$, so $\widetilde{Y}_{(a,b)}$ is not stably equivalent to $\widetilde{\mathbb{P}^1} = S^{2+\alpha}$. However, note that such stable equivalence does hold over the algebraic closure of $k$, therefore, since the suspensions of Tate motives are not equivalent to each other, $S^{2+\alpha}$ is the only possible dimension of a suspension of Tate motive to which $\widetilde{Y}_{(a,b)}$ could be equivalent. Hence, in this case, the affine quadric $Sp(x^2 - ay^2 = b)$ is also not stably equivalent to $\mathbb{G}_m^a$. This also gives the relation in $\mathscr{SH}(k)$

$$\widetilde{U}_{(a,b)} \wedge S^{L_a} \simeq \widetilde{U}_{(b,a)} \wedge S^{L_b}. \qquad \square$$

In fact, the only such Pfister quadrics whose unreduced suspension are invertible are for $n = 1, 2$. (For $n = 1$, $Y_{(a)} = Pr\ (x^2 = ay^2) = Spec(L_a)$, so $\widetilde{Y}_{(a)} = S^{L_a}$.) Suppose $k \subseteq \mathbb{C}$. If the complex realization of $\widetilde{Y}$ is a sphere, then the complex realization of $Y$ itself has to be also a sphere non-equivariantly, after forgetting the $\mathbb{Z}/2$-action. Consider the singular homology of $Y_{(a_1,\dots,a_n)}$ with coefficient $\mathbb{Z}$ after topological realization. Non-equivariantly, this is made up of a copy each of $\Sigma^{2k}H\mathbb{Z}$, where $k$ runs from 0 to $2^{n-1}-1$. For $n>2$, the two copies of $H\mathbb{Z}$ in the two middle dimensions $k=2^{n-2}-1$ and $k=2^{n-2}$ are related by multiplication by 2. So $Y_{(a_1,\dots,a_n)}$ cannot be invertible, since their topological realizations cannot be spheres.

For an arbitrary (perfect) field of characteristic $\neq 2$, this argument can be mimicked using Rost's calculation: recall that the motive of a Pfister quadric $Q$ is Rost motive $M_a$ tensored with the motive of $\mathbb{P}^d$ where $d$ is half the dimension of $Q$, leading to the same conclusion.

In the case of topology, for a compact Lie group $G$, a $G$-equivariant generalized cohomology theory should be graded on the representation ring $RO(G)$ of $G$ to get the complete information [18]. By one-point compactification, $RO(G)$ maps to the Picard group $Pic(\mathscr{SH}(G))$ of the $G$-equivariant stable homotopy category. In the algebraic case, generalized cohomology theories, such as motivic cohomology, are bigraded in a manner compatible with the forgetful functor to $\mathbb{Z}/2$-equivariant cohomology theories, graded on $RO(\mathbb{Z}/2)$, i.e. all dimensions $k+l\alpha$. However, the above examples suggests that to capture full information, generalized algebraic cohomology theories should not only have the two gradings with respect to $S^1$ and $S^\alpha$, but rather with respect to some "motivic representation ring", related to the Picard group $Pic(\mathscr{SH}(k))$.

## 4. $\mathbb{Z}/2$-equivariant and algebraic Hopf invariant one maps

Part of our motivation for understanding invertible objects in the stable $\mathbb{A}^1$-homotopy category stems from trying to formulate the Hopf invariant one problem in it [13,12]. In the $\mathbb{Z}/2$-equivariant topological world, one can state the Hopf invariant one problem as follows. Consider the (free) unit sphere $S(2^n\alpha)$ in the $\mathbb{Z}/2$-representation $2^n\alpha$. One has

$$\tau|_{S(2^n\alpha)} \oplus 1 \cong 2^n\alpha,$$

where $\tau$ denotes the tangent bundle. On the right-hand side here, $2^n\alpha$ denotes the trivial bundle of dimension $2^n\alpha$ on $S(2^n\alpha)$. In the $\mathbb{Z}/2$-equivariant category, to say that the Hopf invariant one property holds in dimension $2^n-1$ is to say that

$$\tau|_{S(2^n\alpha)} \cong 2^n-1.$$

Here, again, $2^n-1$ on the right-hand side denotes the trivial bundle of dimension $2^n-1$. (Note that forgetting the action of $\mathbb{Z}/2$ in the statement above gives the Hopf invariant one property in non-equivariant topology.) If the Hopf invariant one property holds, then in the stable homotopy category, we have that

$$\Sigma^{2^n(\alpha-1)}S(2^n\alpha)_+ \simeq S(2^n\alpha)_+. \tag{4.1}$$

It is well-known that this is true if and only if $n\leqslant 3$. Consider the canonical cofiber sequence

$$S(2^n\alpha)_+ \to S^0 \to S^{2^n\alpha}.$$

Composing the connecting map with the periodicity (4.1), then with the map collapsing $S(2^n\alpha)_+$ gives

$$S^{2^n\alpha-1} \to S(2^n\alpha)_+ \xrightarrow{\simeq} \Sigma^{2^n(\alpha-1)}S(2^n\alpha)_+ \to S^{2^n(\alpha-1)}. \tag{4.2}$$

This is the Hopf invariant one map in the $\mathbb{Z}/2$-equivariant stable homotopy groups of spheres, in dimension $2^n - 1$. These maps are detected in the $Ext^0$ summand of the Real-oriented Adams–Novikov spectral sequence [12]. For further discussion of Hopf invariant one elements in the $\mathbb{Z}/2$-equivariant category, see [10].

One would like to formulate the Hopf invariant one problem in the algebraic stable homotopy category as well. This is done in [13], and we summarize it here as follows. For $a_1, \dots, a_n \in k^{\times}$, consider the Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ given by (1.3). One has the Pfister quadrics

$$X_{(a_1,\dots,a_n)} = Pr\,(\langle\langle a_1, \dots, a_n\rangle\rangle = 0), \tag{4.3}$$

$$Y_{(a_1,\dots,a_n)} = Pr\,(\langle\langle a_1, \dots, a_{n-1}\rangle\rangle = a_n z^2). \tag{4.4}$$

Rost has shown that the motivic homologies of the Pfister quadrics split into a wedge sum of Rost motives $R_{(a_1,\dots,a_n)}$ [27]. Looking with mod 2 coefficients, there is a Rost spectrum $\Theta_{(a_1,\dots,a_n)}$ with the property that

$$H \wedge \Theta_{(a_1,\dots,a_n)} \simeq R_{(a_1,\dots,a_n)},$$

where $H$ denotes the mod 2 motivic cohomology spectrum, and $R_{(a_1,\dots,a_n)}$ is the mod 2 Rost motive (by abuse of notation).

The Rost spectrum $\Theta_{(a_1,\dots,a_n)}$ also has the property that there is a canonical object

$$(S^0)^{\perp}_{(a_1,\dots,a_n)}$$

in $\mathscr{SH}(k)$ associated to $\Theta_{(a_1,\dots,a_n)}$, defined by a cofiber sequence

$$S^{(2^{n-1}-1)(1+\alpha)} \to \Theta_{(a_1,\dots,a_n)} \to (S^0)^{\perp}_{(a_1,\dots,a_n)}, \tag{4.5}$$

see [13]. (Here, to emphasize the analogy between the $\mathbb{Z}/2$-equivariant and algebraic cases, we write $S^1$ for $S^1_{\mathrm{s}}$ and $S^{\alpha}$ for $S^1_{\mathrm{t}}$. So $S^{(2^{n-1}-1)(1+\alpha)} = S^{2^n-2,2^{n-1}-1}$.) When there is no possibility of confusion, we suppress the subscript $(a_1, \dots, a_n)$ in the notation and write just $(S^0)^{\perp}$. In fact, the construction of $\Theta_{(a_1,\dots,a_n)}$ and $(S^0)^{\perp}$ in [13] gives an explicit description of $(S^0)^{\perp}$ as follows. The rational point $(1, 0, \dots, 0, 1)$ of $Y_{(a_1,\dots,a_n,1)}$ gives a splitting

$$S^0 \to Y_+.$$

Then $(S^0)^{\perp}$ is defined by the cofiber sequence

$$S^0 \to (Y\backslash X)_+ \to \Sigma^{1-2^{n-1}(1+\alpha)}(S^0)^{\perp}.$$

Thus, it is

$$\Sigma^{1-2^{n-1}(1+\alpha)} Sp(\langle\langle a_1, \dots, a_n\rangle\rangle = 1)$$

with the basepoint at the rational point $(1, 0, \dots, 0) \in Sp(\langle\langle a_1, \dots, a_n\rangle\rangle = 1)$.

We also recall some basic theory of Pfister quadrics. Let $GW(k)$ denote the Grothendieck–Witt ring of $k$, defined as the group completion of the commutative semiring of isomorphism classes of finite-dimensional quadratic forms on $k$ (recall again $char(k) \neq 2$). Factoring out by the ideal generated by hyperbolic forms, we obtain the Witt ring $W(k)$. The augmentation map $GW(k) \to \mathbb{Z}$ (given by dimension) induces an augmentation map $W(k) \to \mathbb{Z}/2\mathbb{Z}$. Let $I(k)$ be the augmentation ideal of $GW(k)$, and let $I'(k)$ be the augmentation ideal of $W(k)$. There is a map from the mod 2 Milnor $K$-theory

$$K_n^M(k)/2 \to I'(k)^n/I'(k)^{n+1} \tag{4.6}$$

defined by Milnor [19], which sends the generator $a \in K_1^M(k)/2$ to the image of the class $\langle 1\rangle - \langle a\rangle \in I(k)$ in $I'(k)$. The mod 2 Milnor Conjecture, proven by Orlov et al. [24], states that this is an isomorphism (for an outline, see [21]; for a related but different statement, see also [28]). Under the map (4.6), the symbol $(a)$ maps to the Pfister quadric $\langle\langle a\rangle\rangle$ (see [17,21]). Also, recall that if two Pfister forms $\langle\langle a_1, \ldots, a_n\rangle\rangle$ and $\langle\langle a_1', \ldots, a_n'\rangle\rangle$ are the same in $(I'(k))^n/(I'(k))^{n+1}$, then they are in fact equivalent quadratic forms [17, Corollary 10.3.4]. Hence, the Pfister quadric $\langle\langle a_1, \ldots, a_n\rangle\rangle$, and therefore $(S^0)^{\perp}_{(a_1,\ldots,a_n)}$, depends only on the class of $(a_1, \ldots, a_n)$ in the mod 2 Milnor $K$-theory $K_*^M(k)/2$.The map (4.6) maps the symbol $(a_1, \ldots, a_n)$ to the Pfister quadric $\langle\langle a_1, \ldots, a_n\rangle\rangle$.

In particular, suppose that the symbol $(a_1, \ldots, a_n)$ is 0 in $K_n^M(k)/2$. Then the affine quadric $Z = Sp(\langle\langle a_1, \ldots, a_n\rangle\rangle = 1)$ is isomorphic to the quadric

$$Sp(x_1y_1 + \cdots + x_{2^{n-1}}y_{2^{n-1}} = 1).$$

There is a canonical projection $Z \to \mathbb{A}^{2^{n-1}}\backslash\{0\}$, where

$$(x_1, y_1, \ldots, x_{2^{n-1}}, y_{2^{n-1}}) \mapsto (y_1, \ldots, y_{2^{n-1}}).$$

Over each point $(y_1, \ldots, y_{2^{n-1}})$, the fiber is a linear equation in $2^{n-1}$ variables, so it is $\mathbb{A}^{2^{n-1}-1}$. Hence, the projection map is an $\mathbb{A}^{2^{n-1}-1}$-bundle, so $Z$ is $\mathbb{A}^1$-homotopy equivalent to $\mathbb{A}^{2^{n-1}}\backslash\{0\}$. Using the fact that for any $k \geqslant 0$,

$$\mathbb{A}^k\backslash\{0\} \simeq (\mathbb{A}^{k-1}\backslash\{0\}) * (\mathbb{A}\backslash\{0\}) \simeq \Sigma^{1+\alpha}(\mathbb{A}^{k-1}\backslash\{0\})$$

and induction, we get that

$$Z \simeq S^{2^{n-1}(1+\alpha)-1}.$$

Then the cofiber $(S^0)^{\perp}_{(a_1,\ldots,a_n)}$ is $S^0$, and the cofiber sequence (4.5) splits.

On the other hand, for $k \subseteq \mathbb{R}$, in the case that $(a_1, \ldots, a_n) \neq 0$, the complex realization functor to the $\mathbb{Z}/2$-equivariant stable homotopy category takes $(S^0)^{\perp}$ to $S^{2^{n-1}(\alpha-1)}$. In general, $(S^0)^{\perp}$ is the algebraic analogue of the periodicity $S^{2^{n-1}(\alpha-1)}$ in the $\mathbb{Z}/2$-equivariant Hopf invariant one property (4.1). Also, consider the affine quadric

$$U_{(a_1,\ldots,a_n)} = Sp(\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle = a_n).$$

For $(a_1, \ldots, a_n) \neq 0$, it is the algebraic analogue of the free sphere $S(2^{n-1}\alpha)$. Hence, we may state the algebraic Hopf invariant one problem to be the following: for what $a_1, \ldots, a_n \in k^\times$, such that $(a_1, \ldots, a_n) \neq 0$ in $K_*^M(k)/2$, is it true that

$$U_+ \wedge (S^0)^{\perp} \simeq U_+? \tag{4.7}$$

One way in which the Hopf invariant one property can be satisfied is via non-associative division algebras. Let $Q$ be a quadratic form over $k$. We can assume that $Q$ is not isotropic. We have the following definition [13].

**Definition 4.8.** A vector space $V$ over $k$ with quadratic form $Q$ is said to be a non-associative division algebra with norm $Q$ if there is a multiplication

$$\mu : V \otimes_k V \to V$$

a unit $v : k \to V$, and a conjugation $\bar{\ } : V \to V$, such that multiplication is unital, and the following conditions are satisfied:

$$(xy)\overline{y} = xQ(y)$$
$$\overline{xy} = (\overline{y})(\overline{x}).$$

Note that this is actually a division algebra in the usual sense only if the quadratic form $Q$ is not isotropic. As show in [13], if $Q$ is not isotropic, then

$$Q(xy) = Q(x)Q(y).$$

Suppose $Q = \langle\langle a_1, \ldots, a_n \rangle\rangle$ is non-isotropic, with $n \leqslant 3$. Then there is always a non-associative division algebra with norm $Q$, which comes from the usual definitions of complex numbers, quaternions, or octonions. By Hurwitz ([14], see also [26]), these are the only quadratic forms over $k$ which are norms of non-associative division algebras.

Let $U_{(a_1,\ldots,a_n,1)}$ denote the affine quadric

$$Sp(\langle\langle a_1, \ldots, a_n \rangle\rangle = 1)$$

based at the rational point $(1, 0, \ldots, 0)$. So $U_{(a_1,\ldots,a_n,1)} = Y_{(a_1,\ldots,a_n,1)} \backslash X_{(a_1,\ldots,a_n)}$. If $Q = \langle\langle a_1, \ldots, a_{n-1} \rangle\rangle$ is the norm of a non-associative division algebra, then note that

$$(U_{(a_1,\ldots,a_n)})_+ \wedge U_{(a_1,\ldots,b,1)} \cong (U_{(a_1,\ldots,a_n)})_+ \wedge U_{(a_1,\ldots,ba_n,1)}$$

for any $b, a_n \in k^\times$. Namely, given

$$(x_1, \ldots, x_{2^{n-1}}) \in U_{(a_1,\ldots,a_n)}, \ (y_1, \ldots, y_{2^{n-1}}, z_1, \ldots, z_{2^{n-1}}) \in U_{(a_1,\ldots,a_{n-1},b,1)}$$

we map

$$((x_1, \ldots, x_{2^{n-1}}), (y_1, \ldots, y_{2^{n-1}}, z_1, \ldots, z_{2^{n-1}}))$$

to

$$((x_1, \ldots, x_{2^{n-1}}), (y_1, \ldots, y_{2^{n-1}}, (x_1, \ldots, x_{2^{n-1}})(z_1, \ldots, z^{2^{n-1}}))).$$

It is easy to check that this is an isomorphism. For $b=1$, this gives exactly that $(U_{(a_1,\ldots,a_n)})_+$ is $(S^0)^{\perp}_{(a_1,\ldots,a_n)}$-periodic, so Hopf invariant one is satisfied. In fact, this suggests the following more general formulation of the Hopf invariant one problem: for what $a_1, \ldots, a_n, b \in k^\times$, such that $(a_1, \ldots, a_n) \neq 0$ in $K^M_*(k)/2$, is it true that

$$(U_{(a_1,\ldots,a_n)})_+ \wedge (S^0)^{\perp}_{(a_1,\ldots,a_{n-1},b)} \simeq (U_{(a_1,\ldots,a_n)})_+ \wedge (S^0)^{\perp}_{(a_1,\ldots,a_{n-1},ba_n)}? \tag{4.9}$$

We now turn to the question of getting a Hopf invariant one map as an element in the algebraic stable homotopy groups of spheres. In the present notation, Conjecture 1.4 reads:

**Conjecture 4.10.** *For $a_1, \ldots, a_{n-1}, b \in k^\times$, $(S^0)^\perp_{(a_1,\ldots,a_{n-1},b)}$ and $\widetilde{U}_{(a_1,\ldots,a_{n-1},b)}$ are invertible in $\mathscr{SH}(k)$.*

Note that in this paper, we prove this conjecture for $n=1, 2$ (see Section 1 for more discussion). Given the conjecture, if $Q = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ is the norm of a non-associative division algebra, then there is an unstable construction of the algebraic Hopf invariant one map in the algebraic homotopy groups of spheres. Take the multiplication map

$$U_{(a_1,\ldots,a_{n-1},a_n)} \times U_{(a_1,\ldots,a_{n-1},b)} \to U_{(a_1,\ldots,a_{n-1},ba_n)}.$$

Passing to the join gives a map

$$U_{(a_1,\ldots,a_{n-1},a_n)} * U_{(a_1,\ldots,a_{n-1},b)} \to \widetilde{U}_{(a_1,\ldots,a_{n-1},ba_n)}. \tag{4.11}$$

Taking unreduced suspension gives a map

$$\widetilde{U}_{(a_1,\ldots,a_n)} \wedge \widetilde{U}_{(a_1,\ldots,a_{n-1},b)} \to \Sigma\widetilde{U}_{(a_1,\ldots,a_{n-1},ba_n)}. \tag{4.12}$$

Given Conjecture 4.10, this gives a stable Hopf invariant one map of spheres

$$\Sigma^{-1}\widetilde{U}_{(a_1,\ldots,a_n)} \to (\widetilde{U}_{(a_1,\ldots,a_{n-1},b)})^{-1} \wedge (\widetilde{U}_{(a_1,\ldots,a_{n-1},ba_n)}). \tag{4.13}$$

The map (4.13) gives elements of the algebraic stable homotopy groups of spheres in dimensions

$$\Sigma^{-1}\widetilde{U}_{(a_1,\ldots,a_n)} \wedge (\widetilde{U}_{(a_1,\ldots,a_{n-1},b)}) \wedge (\widetilde{U}_{(a_1,\ldots,a_{n-1},ba_n)})^{-1}. \tag{4.14}$$

In particular, if $b=1$, we have a stable map of spheres

$$U_{(a_1,\ldots,a_{n-1},1)} = Sp(\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle = 1) \to S^0. \tag{4.15}$$

There is, however, also a more general stable construction of algebraic Hopf invariant one maps. Consider the cofiber sequence

$$U_+ \to S^0 \to \widetilde{U}$$

in analogy with topology. If Hopf invariant one holds for $(a_1, \ldots, a_n)$, then composing the connecting map with the periodicity and then collapsing gives

$$\Sigma^{-1}\widetilde{U} \wedge (S^0)^\perp_{(a_1,\ldots,b)} \to U_+ \wedge (S^0)^\perp_{(a_1,\ldots,b)} \xrightarrow{\simeq} U_+ \wedge (S^0)^\perp_{(a_1,\ldots,ba_n)} \to (S^0)^\perp_{(a_1,\ldots,ba_n)}.$$

Assuming Conjecture 4.10, we get a map

$$\begin{aligned}\Sigma^{-1}\widetilde{U} &\to Sp(\langle\langle a_1, \ldots, a_{n-1}, b\rangle\rangle = 1) \wedge Sp(\langle\langle a_1, \ldots, a_{n-1}, ba_n\rangle\rangle = 1)^{-1}\\ &= (S^0)^\perp_{(a_1,\ldots,a_{n-1},b)} \wedge ((S^0)^\perp_{(a_1,\ldots,a_{n-1}ba_n)})^{-1}.\end{aligned} \tag{4.16}$$

If $b=1$, the target is $((S^0)^\perp_{(a_1,\ldots,a_n)})^{-1}$, and if $b=a_n$, the target is $(S^0)^\perp_{(a_1,\ldots,a_n)}$. Stably, this is an element of the algebraic stable homotopy groups of spheres in dimension

$$\Sigma^{-1}\widetilde{U}_{(a_1,\ldots,a_n)} \wedge ((S^0)^\perp_{(a_1,\ldots,a_{n-1},b)})^{-1} \wedge (S^0)^\perp_{(a_1,\ldots,a_{n-1},ba_n)}. \tag{4.17}$$

This is the Hopf invariant one element in the algebraic stable homotopy groups. One can of course iterate the periodicity, so the algebraic Hopf invariant one elements for $(a_1, \dots, a_n)$ are in a lattice of dimensions

$$\Sigma^{-1}\widetilde{U}_{(a_1,\dots,a_n)} \wedge \bigwedge_{j=1}^{m}((S^0)^{\perp}_{(a_1,\dots,a_{n-1},b_j)})^{\wedge k_j} \wedge ((S^0)^{\perp}_{(a_1,\dots,a_{n-1}b_j a_n)})^{\wedge -k_j}, \tag{4.18}$$

where $b_1, \dots, b_m \in k^\times$, and $k_1, \dots, k_m$ are integers. In the $\mathbb{Z}/2$-equivariant case, the elements in dimensions (4.18) are 0, at least in the $Ext^0$ part of the Adams–Novikov spectral sequence, if $k_1, \dots, k_m$ are all even [10]. It would be interesting to see if this occurs in the algebraic case also.

It is natural to ask if the dimension (4.14) coincides with any of the dimensions (4.18). Namely, comparing maps (4.13) and (4.16), we can ask: is

$$(S^0)^{\perp}_{(a_1,\dots,a_{n-1},b)} \wedge ((S^0)^{\perp}_{(a_1,\dots,a_{n-1},ba_n)})^{-1} \simeq (\widetilde{U}_{(a_1,\dots,a_{n-1},b)})^{-1} \wedge (\widetilde{U}_{(a_1,\dots,a_{n-1},ba_n)}) \tag{4.19}$$

in $\mathscr{SH}(k)$?

Note that this would follow from the case for $b=1$, namely Conjecture 1.4. This is because Conjecture 1.4 would imply that

$$\widetilde{U}_{(a_1,\dots,a_{n-1},b)} \wedge (S^0)^{\perp}_{(a_1,\dots,a_{n-1},b)} \tag{4.20}$$

is independent of $b$ for all $a_1, \dots, a_{n-1}, b \in k^\times$, so (4.19) holds. Section 1, For more discussion on the connection of these ideas with the Hopf invariant one problem in topology via real and complex realization, the reader is referred to [13].

## 5. Proof of proposition 1.5

Proposition 1.5 states that Conjecture 1.4 is true for $n = 2$. The goal of this section is to prove this proposition. This proof requires algebraic Atiyah duality. Here by algebraic Atiyah duality I mean simply the statement that the map $\lambda_Y$ of (3.22) is an equivalence for every smooth projective variety $Y$. This result is known to experts, but to my knowledge has not yet been published. For completeness, a proof of Atiyah duality in this sense is given in Appendix A.

It should be said that the proof of Proposition 1.5, as written, assumes strong resolution of singularities over $k$, although it is only used in an example, and thus can be surely replaced by direct computation; however, I have not done that at this time.

**Proof of Proposition 1.5.** For fixed $a, b \in k^\times$, $a, b \notin (k^\times)^2$, let $Y$ denote $Pr\ (x^2 - ay^2 = bz^2)$, and let $L_a$ denote the extension field $k[\sqrt{a}]$. For $X \in \mathscr{SH}(k)$, let $DX$ be its Spanier–Whitehead dual. We will show that

$$U_{(a,b,1)} = Sp(\langle\langle a, b\rangle\rangle = 1) \cong \Sigma^{3(1+\alpha)} D(\widetilde{Y}) \tag{5.1}$$

in $\mathscr{SH}(k)$. Then since $\widetilde{Y}$ is invertible, with $(\widetilde{Y})^{-1} = D(\widetilde{Y})$, this gives

$$Sp(\langle\langle a, b\rangle\rangle = 1) \wedge \widetilde{Y} = S^{3(1+\alpha)}. \tag{5.2}$$

But as shown in the proof of Corollary 3.35, we also have

$$\widetilde{Y} = \widetilde{U}_{(a,b)} \wedge \widetilde{Spec(L_a)}.$$

In particular,

$$S^{1+\alpha} = \mathbb{P}^1 = Pr\,(x^2 - ay^2 = z^2) = Sp(x^2 - ay^2 = 1) \wedge \widetilde{Spec(L_a)}.$$

Thus, (5.2) gives

$$\begin{aligned} &Sp(\langle\langle a, b\rangle\rangle = 1) \wedge \widetilde{U}_{(a,b)} \wedge \widetilde{Spec(L_a)} \\ &\quad = \Sigma^{2(1+\alpha)} Sp(x^2 - ay^2 = 1) \wedge \widetilde{Spec(L_a)} \end{aligned}$$

in $\mathscr{SH}(k)$. Since $\widetilde{Spec(L_a)}$ is also invertible, smashing this with $(\widetilde{Spec(L_a)})^{-1}$ gives the proposition.

To prove (5.1), consider the Pfister quadrics

$$\begin{aligned} M &= Y_{(a,b,1)} = Pr\,(x^2 - ay^2 - bz^2 + abt^2 = u^2), \\ X &= X_{(a,b)} = Pr\,(x^2 - ay^2 - bz^2 + abt^2 = 0). \end{aligned}$$

The complement of $X$ in $M$ is $Sp(\langle\langle a, b\rangle\rangle = 1)$. Also, for the inclusion $X \subseteq M$, which is the inclusion of one projective quadric in another, purity holds, i.e. the Thom space of the normal bundle of $X$ in $M$ is the quotient of $M$ by the complement of $X$. This gives a cofiber sequence

$$Sp(\langle\langle a, b\rangle\rangle = 1) = M \backslash X \to M \to X^{\xi}, \tag{5.3}$$

where $\xi$ is the normal bundle of $X$ in $M$. The first map is inclusion.

Recall that in the proof of Proposition 1.2, we showed that for every projective variety $Z$, there is a canonical affine variety $U(Z)$ which is $\mathbb{A}^1$-weak equivalent to $Z$. On an affine variety, a virtual bundle $\zeta$ is always $\zeta' - n$ for some trivial bundle $n$ and actual bundle $\zeta'$. Thus, in the stable homotopy category, we can look at Thom spaces of virtual bundles on projective varieties just like Thom spaces of actual bundles. By Atiyah duality, for smooth variety $X$, the Spanier–Whitehead dual of $X_+$ is

$$D(X_+) = X^{\nu_X},$$

where $\nu_X$ is the stable normal bundle of $X$, a virtual bundle characterized by $\nu_X + \tau_X = 0$. Analogously (see the Remark at the end of the Appendix A), if $\zeta$ is a bundle on $X$, then

$$D(X^{\zeta}) = X^{\nu_X - \zeta}.$$

So the Spanier–Whitehead dual of $X^{\xi}$ is $D(X^{\xi}) = X^{\nu_X - \xi} = X^{\nu_M}$. Taking the dual of (5.3) gives the cofiber sequence

$$X^{\nu_M} \xrightarrow{g} M^{\nu_M} \to D(Sp(\langle\langle a, b\rangle\rangle = 1)). \tag{5.4}$$

Now rewrite the equation defining $M$ as

$$-ay^2 - bz^2 + abt^2 = vw,$$

where $v = x - u$, $w = x + u$. Consider $M' \subset M$, given by the projective equation $v = 0$. If we add the equation $w = 0$, we have $Pr\,(-ay^2 - bz^2 + abt^2 = 0)$. If we add $w \neq 0$, we have

$$Sp(-ay^2 - bz^2 + abt^2 = 0),$$

which is the affine cone on $Pr\ (-ay^2 - bz^2 + abt^2 = 0)$. So

$$M' = Pr\ (-ay^2 - bz^2 + abt^2 = 0)^{\zeta},$$

where $\zeta$ is a line bundle on $Pr\ (-ay^2 - bz^2 + abt^2 = 0)$. The 0-section of $\zeta$ is given by the projective equation $x = u = 0$. But

$$Pr\ (-ay^2 - bz^2 + abt^2 = 0) = Pr\ (t^2 - \frac{1}{a}\, y^2 - \frac{1}{b}\, z^2 = 0).$$

So there is an isomorphism

$$Pr\ (-ay^2 - bz^2 + abt^2 = 0) \xrightarrow{\cong} Pr\ (t^2 - ay^2 - bz^2 = 0) = Y,$$

which takes $t$ to $t$, $y$ to $y/a$, and $z$ to $z/b$. Recall also that the Picard group $Pic(Z)$ of a smooth variety $Z$ is the group of line bundles on $Z$, with a degree map $Pic(Z) \to \mathbb{Z}$ for $Z$ projective (or proper). By [27], $Pic(Y) \cong \mathbb{Z}$, and the degree map on $Pic(Y)$ is $2 : \mathbb{Z} \to \mathbb{Z}$. So a line bundle on $Y$ is determined by its degree. The bundle $\zeta$ has degree 2 (since the degrees of the bundles can be calculated by passage to the algebraic closure of $k$), so it is the canonical line bundle $\gamma_1$, which is determined by the embedding $Y \subset \mathbb{P}^2$. Hence,

$$M' \simeq Y^{\gamma_1}.$$

The complement of $M'$ in $M$ is

$$Sp(-ay^2 - bz^2 + abt^2 = w) \cong \mathbb{A}^3.$$

**Lemma 5.5** (*Uniqueness of 1-point compactification*). *Let $X$, $Y$ be projective varieties over a field $k$. Suppose $U \subset X$ and $V \subset Y$ are smooth open subvarieties. Assuming strong resolution of singularities, an isomorphism $U \to V$ induces, uniquely in the stable $\mathbb{A}^1$-homotopy category, an isomorphism*

$$X/X - U \cong Y/Y - V.$$

**Proof.** Here we need to work in the cd-h category. Starting with either the Nisnievich topology on smooth varieties over $k$, or the cd-h topology on Noetherian schemes over $k$, one obtains the same stable $\mathbb{A}^1$-homotopy category (for these foundational details, we refer the reader to [29]). Now strong resolution of singularities asserts that there exists a smooth projective variety $Z$ with open subvariety $W$ and maps

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

which restrict to isomorphisms

$$\begin{tikzcd} & W \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ U & & V \end{tikzcd}$$

and also preserve the complements. (A classical reference for this use of resolution of singularities is [5, Section 3.2.11].) Moreover, it states that any two such resolutions have a common refinement in the obvious sense. Now restricting the maps $f$, $g$ to the complements of $U$, $V$, $W$, the cd-h square says that the sheaves obtained by pushout with of the inclusion of the exceptional divisor $Z - W$ with $f|_{Z-W}$, resp. $g|_{Z-W}$, are isomorphic to $X$, $Y$, respectively. By transitivity of pushouts (applying pushouts with maps collapsing $X - U$ resp. $Y - V$ to a point), it therefore implies that the above diagrams induce isomorphisms of cd-h sheaves

$$\begin{tikzcd} & Z/(Z-W) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ X/(X-U) & & Y/(Y-V), \end{tikzcd}$$

which pass to isomorphisms in the stable $\mathbb{A}^1$-category. $\square$

In the present case, thus, the quotient $M/M'$ is just $\mathbb{A}^3/(\mathbb{A}^3\backslash\{0\}) \simeq S^{3(1+\alpha)}$. This gives a cofiber sequence

$$Y^{\gamma_1} \xrightarrow{i} M \xrightarrow{p} S^{3(1+\alpha)}.$$

Dualizing this gives the cofiber sequence

$$S^{-3(1+\alpha)} \xrightarrow{Dp} M^{\nu_M} \xrightarrow{Di} DY^{\gamma_1}, \tag{5.6}$$

where the first map is thomification of an inclusion of a point.

On the other hand, by the Segre embedding, we have

$$X \cong Y \times Y. \tag{5.7}$$

Namely, a point $(x_0, y_0, z_0, t_0)$ in $X$ satisfies

$$x_0^2 - ay_0^2 - bz_0^2 + abt_0^2 = 0.$$

Suppose that $z_0 + \sqrt{a}t_0 \neq 0$. Let $N$ denote the norm map from $k[\sqrt{a}]$ to $k$. So we have in $k[\sqrt{a}]$

$$N\left(\frac{x_0 + \sqrt{a}y_0}{z_0 + \sqrt{a}t_0}\right) = b,$$
$$N\left(\frac{x_0 - \sqrt{a}y_0}{z_0 + \sqrt{a}t_0}\right) = b.$$

This gives a map $X \to Y \times Y$ which takes $(x_0, y_0, z_0, t_0)$ to

$$\left(\frac{x_0 + \sqrt{a}y_0}{z_0 + \sqrt{a}t_0}, \ \frac{x_0 - \sqrt{a}y_0}{z_0 + \sqrt{a}t_0}\right)$$

for $z_0 - \sqrt{a}t_0 \neq 0$, and extends uniquely to the remaining points in $Y$. Here, the point on $Y$ corresponding to $u + \sqrt{a}v$ with $N(u + \sqrt{a}v) = b$ is $(u, v, 1) \in Y = Pr\ (u^2 - av^2 = bz^2) \subset \mathbb{P}^2$. This map is an isomorphism, concluding the proof of (5.7).

Now let

$$\Delta : Y \to X \cong Y \times Y \cong X$$

be the inclusion given by $x = 0$. Then the image of $Y$ in $X$ consists of

$$\left(\frac{\sqrt{a}y}{z+\sqrt{a}t}, \frac{-\sqrt{a}y}{z+\sqrt{a}t}\right).$$

So for the two projections $\pi_1, \pi_2 : Y \times Y \to Y$, $\pi_1 \cdot \Delta = Id_Y$, and $\pi_2 \cdot \Delta$ takes $(t, y, z)$ to $(t, -y, z)$. Both these compositions have degree 1. Also, recall the normal bundle $\xi$ of $X$ in $M$. The restriction $\Delta^*\xi$ has degree 2 on $Y$, so it is $\gamma_1$. Thomification of $\Delta$ with respect to $\xi$ gives a map

$$\Delta^\xi : Y^{\gamma_1} \to X^\xi. \tag{5.8}$$

Consider the complement $X \backslash \Delta(Y)$ in $X$. There is the first projection map

$$\pi_1 : X \backslash \Delta(Y) \subset X = Y \times Y \to Y.$$

As shown in the proof of Proposition 1.2, this is an $\mathbb{A}^1$-bundle, thus an $\mathbb{A}^1$-homotopy equivalence. Recall that $DX^\xi = X^{v_M}$, and consider the inclusion $j : X \backslash \Delta(Y) \to X$. Thomification with respect to $v_M$ gives

$$j^{v_M} : (X \backslash \Delta(Y))^{v_M} \to X^{v_M}. \tag{5.9}$$

The cofiber of this map is

$$\Delta(Y)^{v_M+\zeta},$$

where $\zeta$ is the normal bundle of $\Delta(Y)$ in $X$. Since $\xi$ restricts to $\gamma_1$ on $\Delta(Y)$, we have

$$\tau_M - \zeta = (\tau_{\Delta(Y)} + \zeta + \xi) - \zeta = \tau_{\Delta(Y)} + \gamma_1$$

on $\Delta(Y)$. This gives

$$v_M + \zeta = v_{\Delta(Y)} - \gamma_1$$

on $\Delta(Y)$. Therefore, we have the cofiber sequence

$$Y^{v_M} \overset{j^{v_M} \circ (\pi_1^{-1})^{v_M}}{\longrightarrow} X^{v_M} \overset{q}{\to} Y^{v_Y - \gamma_1} = DY^{\gamma_1}. \tag{5.10}$$

(Note that $(\pi_1^{-1})^{v_M}$ exists only in the $\mathbb{A}^1$-homotopy category.) The first map $Y^{v_M} \to X^{v_M}$ is, by construction, homotopic to the Thomification of a map $Y \to X$ of degree $(1, -1)$. The second map $q$ is the Spanier–Whitehead dual of (5.8).

On $X$, the restriction of the tangent bundle $\tau_M$ is the direct sum of the normal bundle $\xi$ of $X$ in $M$, and the tangent bundle $\tau_X$ of $X$, which is again the direct sum of two line bundles. The normal bundle $\xi$ restricts to a bundle of degree 0 via a map of degree $(1, -1)$. Since on $Y \simeq X \backslash \Delta(Y)$, a line bundle depends only on its degree, this is just the trivial line bundle 1. The tangent bundle $\tau_X$ of $X = Y \times Y$ is the direct sum of the two pullbacks of $\gamma_1$ from the two factors. of $\gamma_1$ on the two factors. By the map of degree $(1, -1)$ they restrict to the bundles $\gamma_1$ and $(\gamma_1)^{-1}$ on $Y \simeq X \backslash \Delta(Y)$ by degree calculations. But in the $K$-theory of $Y \simeq X \backslash \Delta(Y)$, we have

$$\gamma_1 + \gamma_1^{-1} = 2.$$

After adding a trivial bundle $n$ to both sides, this implies that in fact $\gamma_1 \oplus \gamma_1^{-1} \cong 2$ stably. Hence, $\tau_M$ restricts to the trivial bundle 3, and $v_M$ restricts to the trivial virtual bundle $-3$ on $X \backslash \Delta(Y) \simeq Y$. So $(X \backslash \Delta(Y))^{v_M} \simeq \Sigma^{-3(1+\alpha)}(X \backslash \Delta(Y))_+$.

Putting together the cofiber sequences (5.4), (5.6) and (5.10), we have the following diagram

$$\begin{tikzcd}
\Sigma^{-3(1+\alpha)}Y_+ \arrow[r, "f"] \arrow[d, "j^{\nu_M}\circ(\pi_1^{-1})^{\nu_M}"'] & S^{-3(1+\alpha)} \arrow[d, "Di"] \\
X^{\nu_M} \arrow[r, "g"] \arrow[d, "D\Delta^{\xi}"'] & M^{\nu_M} \arrow[d, "Dp"'] \\
DY^{\gamma_1} \arrow[r, "="'] & DY^{\gamma_1}
\end{tikzcd} \tag{5.11}$$

where the top horizontal map $f$ is collapse of the base, the middle horizontal map $g$ is the first map of (5.4), and the two vertical compositions are the cofiber sequences (5.6) and (5.10).

We claim that this diagram commutes in the stable homotopy category. For the top square, consider the inclusion

$$\imath : X\backslash\Delta(Y) \overset{\subset}{\to} X \overset{\subset}{\to} M.$$

Recall that the image of $\Delta(Y)$ in $X = Pr\ (x^2 - ay^2 - bz^2 + abt^2 = 0)$ is given by the projective equation $x = 0$, and the image of $X$ in $M = Pr\ (x^2 - ay^2 - bz^2 + abt^2 = u^2)$ is given by the projective equation $u = 0$. Thus, the image of $X\backslash\Delta(Y)$ in $M$ is given by $x \neq 0$, $u = 0$. Recall also that for $M' \subset M$ given by $x - u = 0$, the complement of $M'$ in $M$ is isomorphic to $\mathbb{A}^3$. The image of $X\backslash\Delta(Y)$ in $M$ is contained in $M\backslash M'$, so the inclusion $\imath$ is contractible. Thus, the square

$$\begin{tikzcd}
X\backslash\Delta(Y) \arrow[r] \arrow[d, "\subset"'] & Spec(k) \arrow[d] \\
X \arrow[r, "\subset"] & M
\end{tikzcd}$$

commutes up to homotopy. The top square of the diagram (5.11), up to homotopy, is just thomification of this square with respect to $\nu_M$, so it commutes as well.

For the lower square of (5.11), we dualize it to

$$\begin{tikzcd}
Y^{\gamma_1} \arrow[r, "="] \arrow[d] & Y^{\gamma_1} \arrow[d, "\Delta^{\xi}"] \\
M \arrow[r] & X^{\xi}.
\end{tikzcd}$$

Now the space $Y^{\gamma_1}$ sits in $M$ as the subvariety given by the projective equation $x = u$, and the 0-section is given by the projective equation $x = u = 0$. Thus, on the 0-section, the composition $Y \to M \to X^{\xi}$ maps $Y$ into the 0-section $X$ of $X^{\xi}$ as the projective subvariety given by the equation $x = 0$, i.e. the diagonal $\Delta$. The pullback of $\xi$ to $Y$ with respect to $\Delta$ is $\gamma_1$, so the square commutes. Hence, the lower square of (5.11) commutes as well.

Thus, the stable cofibers of the top and middle horizontal maps of (5.11) coincide in $\mathcal{SH}(k)$. By 5.4, the stable cofiber of the middle horizontal map $g$ is $D(Sp(\langle\langle a, b\rangle\rangle = 1))$. On the other hand, top horizontal map $f$ of (5.11)

$$\Sigma^{-3(1+\alpha)}Y_+ \to S^{-3(1+\alpha)}$$

collapses $Y$ to $Spec(k)$. This has stable cofiber $\Sigma^{-3(1+\alpha)}\widetilde{Y}$. Therefore,

$$D(Sp(\langle\langle a, b\rangle\rangle = 1)) \cong \Sigma^{-3(1+\alpha)}\widetilde{Y}$$

in $\mathscr{SH}(k)$. Dualizing gives (5.1). $\square$

**Corollary 5.12.** *For $a, b \in k^\times$, $(S^0)^\perp_{(a,b)}$ is invertible in $\mathscr{SH}(k)$.*

**Proof.** By definition,

$$(S^0)^\perp_{(a,b)} = \Sigma^{-1-2\alpha}Sp(\langle\langle a, b\rangle\rangle = 1).$$

But $Sp(\langle\langle a, b\rangle\rangle = 1)$ is invertible in $\mathscr{SH}(k)$. $\square$

Thus, in the case $n = 2$, the Hopf invariant one maps are indeed elements in the algebraic stable homotopy groups of spheres. For $(a_1, a_2) \neq 0$ in $K_2^M(k)/2$ and $b = 1$, the unstable construction gives an element

$$Sp(x^2 - a_1y^2 = 0) \to S^0$$

in dimension $\mathbb{G}_m^{a_1}$. The stable construction

$$\Sigma^{-1}\widetilde{U}_{(a_1,a_2)} \to ((S^0)^\perp_{(a_1,a_2)})^{-1}$$

is in dimension

$$\Sigma^{-1}\widetilde{U}_{(a_1,a_2)} \wedge \Sigma^{1+2\alpha}Sp(\langle\langle a_1, a_2\rangle\rangle = 1) = Sp(x^2 - a_1y^2 = 1) = \mathbb{G}_m^{a_1}.$$

So the dimensions of the elements from the stable and unstable constructions coincide. For general $b_1, \ldots, b_m$, the Hopf invariant one elements for $(a_1, a_2)$ are in dimensions

$$\mathbb{G}_m^{a_1} \wedge \bigwedge_{j=1}^{m} Sp(\langle\langle a_1, b_j\rangle\rangle = 1)^{\wedge k_j} \wedge Sp(\langle\langle a_1, b_j a_2\rangle\rangle = 1)^{\wedge -k_j}.$$


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## Appendix A. $\mathbb{A}^1$-duality for smooth projective varieties by Po Hu and Igor Kriz[1]

Let $X$ be a smooth projective variety. The $\mathbb{A}^1$-spectrum $X^{\xi}$ for any virtual bundle $\xi$ on the affinization $X'$, as well as the $X$-normal virtual bundle $v_X$ on $X'$ (and hence the $\mathbb{A}^1$-spectrum $X^{v_X}$) are constructed above in Section 3. The following theorem is a consequence of [4]. For completeness, we give here an elementary proof (a brief outline of the idea of such proof was previously given in [16]).

**Theorem A.1.** *There exist maps in the stable $\mathbb{A}^1$-homotopy category*

$$\eta : S^0 \to X_+ \wedge X^{v_X},$$

$$\varepsilon : X_+ \wedge X^{v_X} \to S^0$$

*such that the compositions*

$$X_+ \xrightarrow{Id\wedge\eta} X_+ \wedge X^{v_X} \wedge X_+ \xrightarrow{\varepsilon\wedge Id} X_+ \tag{A.2}$$

$$X^{v_X} \xrightarrow{Id\wedge\eta} X^{v_X} \wedge X_+ \wedge X^{v_X} \xrightarrow{\varepsilon\wedge Id} X^{v_X} \tag{A.3}$$

*are identities. We also have*

$$X^{v_X} \xrightarrow[\simeq]{\lambda_X} F(X_+, S^0),$$

*where $\lambda_X$ is the map constructed in* (3.22) *above.*

We begin by recalling from Section 3, Claim 2 above that the theorem is valid for $X = \mathbb{P}^n$ and also that for an embedding of smooth projective varieties $X \subset Y$ we have a functorial Gysin isomorphism in the stable $\mathbb{A}^1$-homotopy category

$$X^{v_X^Y} \simeq Y/(Y - X), \tag{A.4}$$

where $v_X^Y$ is the normal bundle of $X$ in $Y$, and in fact more generally for a virtual bundle $\xi$ on $Y'$, a "Thomification" of the Gysin map

$$X^{v_X^Y+\xi} \simeq Y^{\xi}/(Y - X)^{\xi}. \tag{A.5}$$

Now the map $\varepsilon$ was constructed in Section 3 as follows. Collapsing the complement of the diagonal $X$ in $X \times X$, we obtain a map

$$X \times X_+ \to X^{\tau_X}.$$

Thomification of this gives a map

$$X_+ \wedge X^{v_X} \xrightarrow{\alpha} X_+.$$

The map $\varepsilon$ is the composition of $\alpha$ with the collapse map $X_+ \to S^0$.

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Now to construct $\eta$, recall the Thom diagonal

$$X^{\nu_X} \xrightarrow{\theta} X_+ \wedge X^{\nu_X}.$$

Now embed $X \subset \mathbb{P}^n$, and define $\eta$ by composing $\theta$ with the composition

$$S^0 \xrightarrow{q} (\mathbb{P}^n)^{\nu_{\mathbb{P}^n}} \xrightarrow{g} X^{\nu_X}.$$

Here the first map is the dual of the collapse map $\mathbb{P}^n_+ \to S^0$, the second map is given by (A.5).

**Proof of A.2.** It was proven in Section 3 above (in the beginning of the proof of Lemma 3.18) that the composition

$$X_+ \wedge X^{\nu_X} \xrightarrow{Id\wedge\theta} X_+ \wedge X^{\nu_X} \wedge X_+ \xrightarrow{\varepsilon\wedge Id} X_+$$

is $\alpha$. So we need to prove that the composition

$$X_+ \xrightarrow{Id\wedge gq} X_+ \wedge X^{\nu_X} \xrightarrow{\alpha} X_+$$

is the identity. However, we also have the collapse of the complement of the diagonal

$$P : X \subset X \times \mathbb{P}^n$$

and by functoriality of the Gysin map it suffices to prove that

$$X_+ \xrightarrow{Id\wedge q} X_+ \wedge (\mathbb{P}^n)^{\nu_{\mathbb{P}^n}} \xrightarrow{\beta} X_+$$

is the identity where $\beta$ is the Thomification of the collapse of the complement of $P$. Let

$$d : (\mathbb{P}^n)^{\nu_{\mathbb{P}^n}} \to F(\mathbb{P}^n_+, S^0)$$

be the duality map. Then the composition

$$S^0 \xrightarrow{q} (\mathbb{P}^n)^{\nu_{\mathbb{P}^n}} \xrightarrow{d} F(\mathbb{P}^n_+, S^0)$$

is the composition with collapse. So it suffices to prove the commutativity of the diagram

$$\begin{tikzcd}
(\mathbb{P}^n)^{\nu_{\mathbb{P}^n}} \wedge X_+ \arrow[r, "\beta"] \arrow[d, "d\wedge Id"'] & X_+ \\
F(\mathbb{P}^n_+, S^0) \wedge X_+ \arrow[r, "Id\wedge P"] & F(\mathbb{P}^n_+, S^0) \wedge \mathbb{P}^n_+ \wedge X_+. \arrow[u, "ev\wedge Id"']
\end{tikzcd}$$

(precomposing with $q \wedge Id$ and going around the three sides of the diagram gives the identity). But using the compatibility between $\varepsilon$ and the evaluation map for $\mathbb{P}^n$, this diagram is equivalent to

$$\begin{tikzcd}
(\mathbb{P}^n)^{\nu_{\mathbb{P}^n}} \wedge X_+ \arrow[r, "\beta"] \arrow[d, "Id\wedge P"'] & X_+ \\
(\mathbb{P}^n)^{\nu_{\mathbb{P}^n}} \wedge \mathbb{P}^n_+ \wedge X_+ \arrow[ur, "\varepsilon\wedge Id"'] &
\end{tikzcd}$$

which is naturality of the Thomified Gysin map followed by collapse map from $\mathbb{P}^n_+ \wedge X_+$ to $X_+$ (naturality with respect to the inclusion/pullback

$$\begin{tikzcd} X \arrow[r, "P"] \arrow[d, "P"'] & \mathbb{P}^n \times X \arrow[d, "Id\times P"] \\ \mathbb{P}^n \times X \arrow[r, "\Delta\times Id"] & \mathbb{P}^n \times \mathbb{P}^n \times X \end{tikzcd}$$

where the vertical inclusions are the ones whose complement is collapsed). □

**Proof of A.3.** Thomification of the proof of (A.2); in fact, for a general bundle $\xi$ on $X$,

$$X^\xi \xrightarrow{Id\wedge\eta^\xi} X^\xi \wedge X^{\nu_X-\xi} \wedge X^\xi \xrightarrow{\varepsilon^\xi\wedge Id} X^\xi \tag{A.6}$$

is the identity by the same argument.

In more detail, consider the diagram

$$\begin{tikzcd} X^\xi \arrow[r, "Id\wedge gq"] \arrow[dr, "Id\wedge\eta^\xi"'] & X^\xi \wedge X^{\nu_X} \arrow[rr, "\alpha^\xi"] \arrow[d] & & X^\xi \arrow[d, "\theta"'] \arrow[drr, "Id"] & & \\ & X^\xi \wedge X^{\nu_X-\xi} \wedge X^\xi \arrow[rr] & & X_+ \wedge X^\xi \arrow[rr, "coll."'] & & X^\xi. \end{tikzcd} \tag{A.7}$$

Here the middle square of this diagram is again naturality of the Thomified Gysin map with respect to pullback. Consequently, it suffices to prove again that the top row of the diagram (A.7) is the identity. But just as above, we may replace this in turn by the map

$$X^\xi \xrightarrow{Id\wedge q} X^\xi \wedge (\mathbb{P}^n)^{\nu_{\mathbb{P}^n}} \xrightarrow{\beta^\xi} X^\xi. \tag{A.8}$$

Showing that this is the identity is now done by precisely the argument as in the case $\xi=0$, but thomified with respect to $\xi$. □

To conclude the proof of the Theorem, we use the following formal

**Lemma A.9** (*Dold and Puppe [6]*). *Suppose we have maps*

$$\eta : S^0 \to X \wedge Y, \quad X \wedge Y \to S^0$$

*so that the compositions*

$$X \xrightarrow{Id\wedge\eta} X \wedge Y \wedge X \xrightarrow{\varepsilon\wedge Id} X \tag{A.10}$$

$$Y \xrightarrow{Id\wedge\eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon\wedge Id} Y \tag{A.11}$$

*are equal to the identity. Then we have equivalences*

$$X \simeq F(Y, S^0), \quad Y \simeq F(X, S^0).$$

**Proof.** Note that $\varepsilon$ gives canonical maps

$$d_Y : Y \to F(X, S^0) \tag{A.12}$$

$$d_X : X \to F(Y, S^0). \tag{A.13}$$

Now let $e_Y$ be the composition

$$F(X, S^0) \xrightarrow{Id\wedge\eta} F(X, S^0) \wedge X \wedge Y \xrightarrow{ev\wedge Y} Y$$

and define $e_Y$ similarly. Now consider the commutative diagram

$$\begin{tikzcd}
X \wedge F(X, S^0) \arrow[r, "X\wedge Id\wedge\eta"] \arrow[ddr, "T"'] & X \wedge F(X, S^0) \wedge X \wedge Y \arrow[r, "Id\wedge ev\wedge Id"] \arrow[d, "T\wedge T"] & X \wedge Y \arrow[d, "Id\wedge d_Y"] \\
& F(X, S^0) \wedge X \wedge Y \wedge X \arrow[d, "\varepsilon"] & X \wedge F(X, S^0) \arrow[d, "ev"] \\
& F(X, S^0) \wedge X \arrow[r, "ev"'] & S^0.
\end{tikzcd}$$

Commutativity is by definition, but the adjoint says that the composition

$$F(X, S^0) \xrightarrow{e_Y} Y \xrightarrow{d_Y} F(X, S^0)$$

is the identity, so $d_Y$ is a retraction, and similarly $d_X$. But now the evaluation $\varepsilon$ factors (by definition)

$$\begin{tikzcd}
Y \wedge X \arrow[r, "\varepsilon"] \arrow[d, "d_Y\wedge Id"'] & S^0 \\
F(X, S^0) \wedge X \arrow[ur, "ev"'] &
\end{tikzcd}$$

so (A.11) (which is the identity) factors as

$$Y \xrightarrow{d_Y} F(X, S^0) \xrightarrow{Id\wedge\eta} F(X, S^0) \wedge X \wedge Y \xrightarrow{ev\wedge Id} Y$$

so $d_Y$ is also a retract, hence an equivalence. Similarly $d_X$. □

**Remark 1.** By what we have proved, we may replace, in (A.2), $X^{\nu_X}$ by $F(X_+, S^0)$ and $\varepsilon$ by evaluation. Then (A.2) is the adjoint form of saying that the composition

$$S^0 \xrightarrow{\eta} F(X_+, S^0) \wedge X_+ \xrightarrow{\wedge} F(X_+, X_+)$$

is the unit, which is the strong dualizability condition. Hence, we have proved that $X_+$ is strongly dualizable. More generally, since all of our arguments can be thomified, we have proved that for any virtual bundle $\xi$ on $X'$, $X^\xi$ is strongly dualizable, with dual $X^{\nu_X-\xi}$.

**Remark 2.** It is worth mentioning that this treatment of duality works for smooth projective schemes over a Noetherian base scheme $S$.

## References

[3] A.K. Bousfield, E.M. Friedlander, Homotopy theory of $\Gamma$-spaces, spectra, and bisimplicial sets, Lecture Notes in Mathematics, vol. 658, Springer, Berlin, 1978, pp. 80–130.
[4] P. Deligne, Voevodsky's lectures on cross functors, preprint, Fall 2001, updated 17 January 2002.
[5] P. Deligne, Theéorie de Hodge II, Inst. Hautes Études Sci. Publ. Math. 40(1071) 5–57.
[6] A. Dold, D. Puppe, Duality, trace and transfer, Trudy Mat. Inst. Steklov 154 (1983) 81–97.
[7] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. Math. 79 (1964) 109–326.
[10] P. Hu, The $Ext^0$-term of the Real-oriented Adams Novikov spectral sequence, in homotopy methods in algebraic topology (Boulder, CO 1999), Contemp. Math. 271 (1999) 141–153.
[11] P. Hu, $S$-modules in the category of schemes, Mem. AMS 161 (767) (2003).
[12] P. Hu, I. Kriz, Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence, Topology 40 (2) (2001) 317–399.
[13] P. Hu, I. Kriz, Some remarks on Real and algebraic cobordism, $K$-theory 22 (4) (2001) 335–366.
[14] A. Hurwitz, Über die Komposition der quadratischen Formen, Math. Ann. 88 (1923) 1–25.
[15] J.F. Jardine, Motivic symmetric spectra, Doc. Math. 5 (2000) 445–553.
[16] B. Kahn, Le conjecture de Milnor (d'après V. Voevodsky) Séminaire Bourbaki, vol. 1996/97, Astérisque 245 (5) (1997) 379–418 (Exp. 834).
[17] T.Y. Lam, The Algebraic Theory of Quadratic Forms, Benjamin/Commings, New York, 1973.
[18] J.P. May, Picard groups, Grothendieck rings, and Burnside rings of categories, Adv. Math. 163 (1) (2001) 1–16.
[19] J. Milnor, Algebraic $K$-theory and quadratic forms, Invent. Math. 9 (1970) 318–344.
[21] F. Morel, Suite spectrale d'Adams et invariants cohomologiques des formes quadratiques, C. R. Acad. Sci. Paris Ser. I Math. 328 (11) (1999) 963–968.
[22] F. Morel, V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, Pub. Math. IHES (90) (1999) 45–143.
[23] Y. Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequence in algebraic $K$-theory, Algebraic $K$-theory: Connections with Geometry and Topology, Kluwer Academic Publishers, Dordrecht, 1989, pp. 241–342.
[24] D. Orlov, A. Vishik, V. Voevodsky, An exact sequence for Milnor's $K$-theory with applications to quadratic forms, preprint, AG/0101023.
[26] M. Rost, On the dimension of a composition algebra, Doc. Math. 1 (1996) 209–214.
[27] M. Rost, The motive of a Pfister form, preprint, 1998.
[28] V. Voevodsky, The Milnor conjecture, preprint, 1996, UIUC $K$-theory archive/0170.
[29] V. Voevodsky, Unstable motovic homotopy categories in Nisnevich and cdh-topologies, preprint, 2000.
[30] V. Voevodsky, Motivic cohomology with $\mathbb{Z}/2$-coefficients, Pub. Math. IHES 98 (2003) 59–104.